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DEPARTMENT OF ECONOMICS
RESEARCH MEMORANDUM



**WARRANT PRICING: A REVIEW OF
THEORETICAL AND EMPIRICAL RESEARCH**

Drs. C.H. Veld

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WARRANT PRICING: A REVIEW OF THEORETICAL AND EMPIRICAL RESEARCH. 1 2

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Notes

1. Introduction.

From January 1, 1976 until June 30, 1990, 35 Dutch companies made 41 issues in which 46 different warrants³, listed on the Amsterdam Stock Exchange (ASE) were involved⁴. An interesting question is how these warrants are priced⁵. In order to be able to answer this question, the problem of warrant pricing should be considered. Therefore we will present a review of the theoretical and empirical analysis that has been made on this subject. After this review is completed, a methodology will be presented for a study after the pricing of Dutch warrants. This paper is constructed as follows.

In section 2 option pricing models will be considered. These models fall into one of two categories: (1) ad hoc models and (2) the Black/Scholes (1973) model and its subsequent modifications. In section 3 the Kassouf (1969) model will be

discussed as an example of an ad hoc model. In section 4, the well-known model of Black and Scholes (1973) is discussed. In this section also the complications of the Black/Scholes (1973) model for the valuation of warrants are considered. These complications are compared with the complications for call-options. The latter term reflects in this paper call-options that are traded on an Official Options Exchange. In section 5, the parameter of the Black/Scholes (1973) model, that is most difficult to estimate, i.e. the standard deviation of the rates of return on the underlying stock, will be considered. This is followed by a discussion of alternative option pricing models, based on the Black/Scholes (1973) model in section 6. These alternative models all try to relax one or more of the restrictive assumptions underlying the Black/Scholes (1973) model. In section 7 the most important difference between warrants and call-options will be discussed. This is that exercise of a warrant, in contrary to exercise of a call-option, leads to a creation of new shares. This causes an additional valuation-problem, the so-called 'dilution problem'. In section 8 attention will be paid to empirical tests of option pricing models for the valuation of warrants. After having reviewed the theory of option- and warrant-pricing we will be able to construct a methodology for the pricing of warrants. This will be presented in section 9. The paper will finish with section 10 in which the most important elements of this study will be summarized.

2. Option pricing models.

Smith (1976) argues that option pricing models fall into one of two categories:

- 1) ad hoc models;
- 2) equilibrium models.

According to Smith (1976) the ad hoc models generally appear in the non-academic literature and are the result of casual empiricism or curve-fitting exercises - not of maximizing

behaviour on the part of market participants. Examples of ad hoc models include Shelton (1967b), Kassouf (1968 and 1969) and Van Horne (1969).

Although Smith (1976) does not explicitly define 'equilibrium models', the context of his remark indicates that these models include the Black and Scholes (1973) option pricing model, hereafter to be referred to as the B/S-model, and its subsequent modifications. Leaving the question undiscussed whether the B/S-model can be classified as an equilibrium model⁶ we argue that the classification of Smith (1976) is useful in a practical manner. Therefore we will make a difference between ad hoc models on one side and the B/S-model and the subsequent developed models on the other side. By means of example we will pay attention to one of the ad hoc models in section 3, i.e. the Kassouf (1969) model. In sections 4 and 5 we will pay attention to the B/S-model and the subsequent developed models.

3. Example of an ad hoc model: The Kassouf model.

Ad hoc models generally have the form of "multiple regression warrant valuation models". Noreen (1982) argues that perhaps the most sophisticated model of this genre is the Kassouf (1969) model, because it does not assume a linear relationship between warrant prices and common stock prices. Therefore we will present Kassouf's model as an example of an ad hoc model based on multiple regression:

$$W = \{ [(S/X)^z + 1]^{1/z} - 1 \} * X \quad (1)$$

where:

W = model price of a warrant;
 S = price of the underlying common stock;
 X = exercise price of the warrant;
 z = a parameter of the specific warrant.

Kassouf (1969) further suggests that a warrant's "z" may be approximated by a linear function of:

- the time to expiration of a warrant;
- the dividend yield on the underlying common stock;
- the ratio of the number of shares to be issued upon warrant-exercise and the number of shares outstanding;
- x - the slope of the least squares line fitted to the logs of the monthly mean price of common stock for the previous eleven months;
- x - the standard deviation of logs of monthly mean price of common stock for the previous eleven months;
- the ratio of the stock price and the exercise price;
- the exercise price.

In section 8.2 a test of the Kassouf (1969) model will be discussed which is performed by Noreen (1982).

4. The application of the Black/Scholes model for the valuation of warrants.

4.1. The Black/Scholes model.

In 1973 Black and Scholes published their well-known option pricing model. This model is derived under the following assumptions⁷:

- 1) the stock pays no dividends or other distributions;
- 2) the option is "European", that is, it can only be exercised at maturity;
- 3) the short-term interest rate is known and is constant through time;
- 4) the stock price follows a random walk in continuous time with a variance rate proportional to the square of the stock price; thus the distribution of possible stock prices at the end of any finite interval is lognormal; the variance rate of the return on the stock is constant;
- 5) there are no transaction costs in buying or selling the stock or the option;
- 6) there are no penalties to short selling; a seller who does not own a security will simply accept the price of

the security from a buyer, and will agree to settle with the buyer on some future date by paying him an amount equal to the price of the security on that date;

- 7) it is possible to borrow any fraction of the price of a security to buy it or to hold it, at the short-term interest rate.

Black and Scholes (1973) demonstrate that it is possible to create a riskless hedge by forming a portfolio containing stock and European call-options. The return on this portfolio must be the riskless rate of return, because otherwise arbitrage opportunities exist. Using advanced stochastic calculus Black and Scholes (1973) derive an equation for the calculation of the value of a call-option (C) consisting of four observable variables: the price of the underlying stock (S); the exercise price (X); the time to maturity (T-t), from now on to be referred to as the maturity; and the riskless interest rate⁸ (r_f), and one variable that is not observable, the volatility of the instantaneous rate of return on the stock, from now on to be referred to as the volatility. In the Black-Scholes model the volatility is defined as the instantaneous standard deviation of the stock's distribution of rates of return (per year). From now on we will refer to this as the standard deviation (σ), which is assumed constant. The Black-Scholes equation can be presented as follows:

$$C = SN(d_1) - Xe^{-r_f(T-t)} N(d_2) \quad (2)$$

where:

$N(.)$ = cumulative standard normal distribution;

$$d_1 = \frac{\ln(S/X) + (r_f + \sigma^2/2)(T-t)}{\sigma\sqrt{(T-t)}}$$

$$d_2 = d_1 - \sigma\sqrt{(T-t)};$$

Although, according to Black (1989), the original intention was to construct a formula for the valuation of warrants, the final equation turned out to be more useful for options than for warrants. In the next sub-section we will discuss the problems that exist if the Black/Scholes (1973) model (from

now on B/S-model) is to be applied for warrants.

4.2. Complications in the use of the Black/Scholes model for the valuation of warrants.

In this section the complications that exist for warrants will be compared with the complications that exist for call-options. These complications will be discussed in relation to the assumptions underlying the B/S-model. We notice that a number of complications are related to the fact that warrants have maturities of several years. In this context it is interesting to notice that since 1986 also call-options with an initial maturity of 5 years are traded on the European Options Exchange in Amsterdam. These options are written on the shares of 5 Dutch multinationals. Recently the example of the EOE was followed by the Chicago Board of Options Exchange (CBOE). From October 5, 1990 call-options with an initial maturity of 2 years, written on the shares of 14 US multinationals, are traded on the CBOE. These kind of call-options will from now on be referred to as long term call-options. Of course the complications related to the long maturity of warrants, are also relevant for the long term call-options.

ad. assumption 1) No dividend payments on the underlying stock.

Warrants, like call-options, are generally not protected against payments of cash-dividends. Over a maturity of several years the impact of anticipated dividends and uncertainty in the dividend payments may be important for the value of warrants and long term call-options. We notice that both warrants and call-options are generally protected against payments of (large) stock-dividends in the form of a so-called 'anti-dilution clause'.

ad. assumption 2) The option is of the "European type".

The second assumption of the B/S-model is that the warrant

can only be exercised at maturity. However, most warrants are of the "semi-american type", that is they can be exercised from a few months (or years) after the issue is completed until the end of their maturity. The same problem occurs for long term call-options which are generally of the American type⁹. Especially in case large (cash-)dividends are paid before the maturity, early exercise may become an important issue.

ad. assumption 3) The interest rate is constant.

In practice the riskless rate of interest will be expected to fluctuate over a period of several years. Fortunately the expected changes are in practice indicated by the 'term structure of interest rates'. Therefore the riskless interest rate can be estimated as the return on a government bond that matures at the same time as the warrant. Because of the availability of returns on government bonds, the fact that the riskless interest rate is not constant, will not be an important problem.

ad. assumption 4) The variance rate of the return on the stock is constant.

A matter closely related to the foregoing is that over a period of several years the variance rate of return on stock, in the B/S-model defined as the standard deviation, may be expected to change substantially. Hull (1989, page 304) argues that in case the standard deviation is expected to rise steadily from 20% to 30% during the remaining maturity, it would be appropriate to use a standard deviation of 25% when valuing the warrant. We notice however that such known changes in the standard deviation will hardly ever be available. Especially in the case of warrants, which generally have maturities of several years, it will be very difficult to make an estimation of the average standard deviation that will be relevant during the maturity of the warrant.

We notice that especially these four assumptions have received much attention in the option pricing literature. A number of models, based on the B/S-model have been developed in which one or more of these assumptions are relaxed. In section 5 we will discuss some of these models.

ad. assumption 5) No transaction costs.

Transaction costs may be important if small numbers of call-options or warrants are to be bought and later exercised. By means of illustration we have calculated the transaction costs to be paid by a private person for a fictive call-option, an otherwise identical warrant and a share of common stock. The results of this calculation are presented in appendix A. From this appendix we conclude that the transaction costs per share are (much) higher in case call-options and warrants are to be bought and later exercised, than in case shares of common stock are to be bought directly. This is especially true if small amounts of shares are to be purchased (i.e. ≤ 100). It is interesting to notice that the transaction costs for warrants are lower than the transaction costs for call-options, therefore this assumption is more severe for call-options than it is for warrants.

ad. assumption 6) No penalties to short selling.

In case an investor subjectively believes that a specific EOE call-option, which is not in his possession, is overvalued, he can simply create a short position in call-options in order to take advantage of this alleged overvaluation. According to information provided to us by the Amsterdam Stock Exchange, such opportunities also exist in the case of warrants but in a specific way. On the Amsterdam Stock Exchange a short position in warrants can only be maintained from the day the short position is taken until the "fixed settlement date", which is the date at which the 'short securities' must be delivered. This fixed settlement date was until February 5, 1990, the date 10 trading days after the

short position was taken. From February 5, 1990 the fixed settlement date is determined at only 7 calendar days after the date the short position is taken. Although the rules of the Amsterdam Stock Exchange do not forbid or exclude the maintaining of a short position during the period described above, it is possible for the mediating bank and/or stock-broker to issue restrictive rules. Therefore the period of 10 trading days (until February 5, 1990) or 7 calendar days (from February 5, 1990) can be considered as the maximum period for holding a short position in warrants. To summarize we argue that, although the opportunities to create short positions using warrants are restricted to a limited number of days, it is certainly true that instantaneously riskless hedge-portfolio's in the sense of Black and Scholes can be formed.

ad. assumption 7) Borrowing at the short-term interest rate is possible.

This assumption is not different for call-options and warrants and will therefore be left undiscussed.

In addition to the earlier mentioned violations of assumptions underlying the B/S-model, we mention the fact that warrants, unlike call-options, often have special conditions included in the warrant-agreement. This is due to the fact that call-options are traded on an official Options Exchange. Therefore they are subjected to standardized conditions. Warrants are traded on the Stock Exchange where warrant-issuers have more flexibility in determining the conditions. The most occurring special possibilities will be discussed below.

Black and Scholes (1973) mention the following special conditions:

- the inclusion of a 'step-up exercise price', these warrants have an exercise price that increases on specific dates in the future, e.g. the exercise price of the warrants issued

on July 31, 1987 by the Dutch company European Development Capital Corporation is:

- \$ 11.-- from September 1, 1988 until August 31, 1989,
- \$ 12.-- from September 1, 1989 until August 31, 1990 and
- \$ 13.-- from September 1, 1990 until August 31, 1991.

- the right for the warrant holder to pay the exercise price using bonds of the company at face value, even though they may at the time be selling at a discount¹⁰.

In addition we mention a number of other special conditions that might also be relevant:

- some warrants are callable, that is the company has the right to call these warrants before their scheduled maturity at certain conditions; if such a warrant is called, the warrant holder has the choice between exercising his warrant or letting it expire;
- the inclusion of the companies' right to reduce the exercise price on a temporal or definitive basis;
- according to Longstaff (1990) the conditions of many US-warrants include the right for the company to extend the warrant's maturity¹¹.

The last important difference between call-options and warrants we mention is that when call-options are exercised this only leads to an exchange of existing shares from one market participant to another. In case warrants are exercised, new shares are created, and the exercise price paid for them becomes part of the assets of the firm. This is known as the 'dilution problem'. We will postpone the discussion on this subject until section 7. This is due to the fact that this discussion requires knowledge about the volatility parameter of the option pricing model (to be discussed in section 5) and the binomial model (to be discussed in section 6). Therefore we will first discuss some topics derived in Option Pricing Theory in the next sections.

5. The estimation of the standard deviation.

5.1. The historical standard deviation.

The traditional way to estimate the risk of common stocks is to calculate a historical standard deviation by using statistics based on a time series of realized rates of return. The implicit assumption is that past experience will repeat itself and that the ex post (historical) standard deviation is a good estimate of the future one.

Hull (1989, page 88) argues that in order to estimate the historical standard deviation of the stock price empirically, the stock price is observed at fixed intervals in time, e.g. every day. In equation (3) the estimation of the historical standard deviation is presented:

$$\sigma_{\text{hist}} = s/\sqrt{\tau} \quad (3)$$

where:

σ_{hist} = estimation of the historical standard deviation;

τ = length of time interval in years;

s = an unbiased estimate of the standard deviation of u_i ;

u_i = continuously compounded return (not annualized) in the i th interval for $i = 1, 2, \dots, n$, this is equal to $\ln(S_i/S_{i-1})$;

S_i = stock price at the end of i th interval.

The unbiased estimate, s , of the standard deviation of the u_i 's is given by:

$$s = \sqrt{\left\{ \frac{1}{n-1} \sum_{i=1}^n u_i^2 - \frac{1}{n(n-1)} \left(\sum_{i=1}^n u_i \right)^2 \right\}} \quad (4)$$

The analysis above assumes that the stock pays no dividends. Hull (1989, page 89) presents the following formula for the return, u_i , during a time interval that includes an ex-dividend day¹²:

$$u_i = \ln(S_i + D/S_{i-1}) \quad (5)$$

where:

D = amount of the dividend.

Two remarks must be made concerning the estimation of the historical standard deviation.

With regard to the choice of an appropriate value for n , Hull (1989, page 89) argues that more data generally lead to more accuracy. However he argues, on the other side the standard deviation changes over time and data that are too old may not be relevant for predicting the future. Hull (1989, page 89) and Jarrow and Rudd (1983, page 137) suggest that the use of stock prices from daily data over the most recent 90 to 180 days might be a suitable compromise.

Another important issue is whether time should be measured in calendar days or trading days when volatility parameters are being estimated and used. Based on research by Fama (1965) and French (1980), Hull (1989, pages 89 and 122) concludes that trading days should be used, because volatility proved to be far larger when the exchange is open than when it is closed.

5.2. The implied standard deviation.

5.2.1. Introduction.

Latané and Rendleman (1976) were the first to suggest the estimation of an ex ante standard deviation from option prices by using the B/S-model. It is assumed that the B/S-model and the required assumptions for its derivation are valid and that stock- and option markets are efficient. Under such conditions, by equating the model value of an option to its market price, the implied standard deviation (ISD) can be calculated. The following example may clarify the concept of the ISD.

Example 1:

On January 22, 1990 call-options Philips were outstanding with the following parameters:

- Exercise price $(X) = f 40.--$
- Price of the underlying common stock $(S) = f 44.--$
- Time to maturity $(T-t) = 179 \text{ days} = 0.492 \text{ years}$
- Riskless interest rate $(r_f) = 8\% = 0.08$
- Standard deviation calculated on the basis of historical¹³ data $(\sigma) = 21.7\% = 0.217$

Using these parameters a B/S-value of $f 6.19$ results. The market price of the option was $f 6.30$. If a standard deviation of 23.1% would have been assumed, the model price would have equated the market price. Therefore the standard deviation of 0.231 is referred to as the 'implied standard deviation'.

If ISDs are calculated for options written on the same stock but having other exercise prices and maturities, theoretically no differences would be expected. This is due to the assumptions of model validity and market efficiency. Latané and Rendleman (1976) argue that this will as a practical matter not be the case because some options are more dependent upon a precise specification of the standard deviation than others. They argue that for options such as those which are in-the-money¹⁴ with little time to maturity, an exact specification of the standard deviation hardly matters. However, for other types of options it may be very important. Therefore Latané and Rendleman (1976) argue that some kind of weighing scheme must be developed in order to come to a weighted ISD. This brings us to a question that has received much attention in option pricing literature, after the appearance of the paper written by Latané and Rendleman (1976)¹⁵: can the future standard deviation best be predicted by the historical standard deviation or by the implied standard deviation resulting from a specific weighing scheme?

This question will be discussed in the next sub-section.

future return variability than the HSD was.

ii) The Beckers (1981) study.

Beckers (1981) uses the B/S-model with an ad hoc dividend correction in order to calculate ISDs from options traded on the CBOE and the NYSE in the 75-day trading interval between October 13, 1975 and January 23, 1976. In contrast to Latané and Rendleman (1976) only ISDs with the same maturity are put together in weighted ISDs, the reason for this will be discussed in sub-section 5.2.3. Beckers (1981) uses the following weighing schemes:

- 1) the WISD, as suggested by Latané and Rendleman (1976);
- 2) the BISD, a weighing scheme developed by Beckers himself, that puts more weight on options that are highly sensitive upon an exact specification of the standard deviation¹⁹;
- 3) the AMISD, which is simply using the ISD for the most sensitive option; although this option is generally slightly out-of-the-money, Beckers (1981) refers to the ISD derived from this option as the at-the-money ISD (AMISD).

From his first test Beckers (1981) concludes that the BISD tends to outperform the WISD and that both are inferior to the AMISD. Beckers (1981) also concludes that ISDs are volatile over time, which may be due to an overreaction of the market on new information or to the existence of a bid-ask spread. Due to the last matter, the last trade (which results in the closing price) may have been executed at the bid price, the ask price or some price in between.

In his second test, over the period from April 28, 1975 to July 22, 1977 Beckers (1981) uses a five-day arithmetic average ISD for the BISD and the AMISD. The WISD is omitted from this test. Using the methodology from scheme 1 also the HSD and the FBISD are compared with the actual (realized) standard deviation. The FBISD (the Fisher Black Standard Deviation) is an estimation of the standard deviation that is sold by Fisher Black's option service to option traders.

These estimates rely heavily on ISDs but include also additional information such as information derived from the HSD. From his second test Beckers (1981) concludes that the worst predictor is in practically all cases the HSD. He also concludes that the FBISD is the best predictor, followed by the AMISD and the BISD. It is interesting to notice that, in a later study, Gemmill (1986) also finds that there is relevant information in both historical and implied standard deviations. However, Gemmill (1986) does not succeed in combining this information in a superior forecast.

Finally Beckers (1981) also tests transaction data²⁰ against closing price data. This is only done for a small interval. Transaction data are believed to outperform closing price data, because of the problems of non-simultaneity of stock- and option prices and of the existence of a bid-ask spread. On the basis of his limited data-set Beckers (1981) proves that transaction data outperform closing price data.

In a larger study on this subject, Brenner and Galai (1984) also reach the conclusion that transaction data are superior to closing price data.

5.2.3. The "term structure of volatility".

In section 5.2.2. it has already been noticed that Latané and Rendleman (1976) calculate weighted ISDs for all options that were traded on a specific date, while Beckers (1981) argues that weighted ISDs should be calculated for each maturity. Beckers (1981) states that a distinction should be made between options written on the same stock but having different maturities since they have different time horizons. He argues that the market's perception of the stock's standard deviation over the remaining life of the option could therefore differ depending upon the time to maturity. Brenner and Subrahmanyam (1988) agree with Beckers that different perceptions exist on short-run versus long-run standard deviation. They refer to this phenomenon as "the term structure of volatility".

Kemna (1988, pages 104-111) tests fourteen EOE options for three different maturities over the period from August 13, 1984 until December 28, 1984. She concludes that the near-term average ISD was always significantly different from the middle- and long-term average ISD and that the middle-term average ISD was in 11 out of 14 cases significantly different from the long-term ISD. She also concludes for 12 out of 14 cases that an increase in the time to maturity leads to a decrease in the average ISD.

5.2.4. Exercise price biases.

There exists little consensus over the nature of the exercise price bias of the B/S-model. Rubinstein (1985) reports several studies on the direction of the exercise price bias. He first reports from a study by Black (1975), who makes a research in the early years of trading in CBOE-options. From Black's study it can be concluded that ISDs, derived from the B/S-model, tended to be higher for out-of-the-money options in relation to in-the-money options. MacBeth and Merville (1979) who make a research under CBOE-options traded from December 31, 1975 to December 31, 1976 conclude that the B/S-model produces high ISDs for in-the-money options and low ISDs for out-of-the-money options. Rubinstein (1985) uses transactions data from the Berkeley Option Data Base for two time-intervals, from August 23, 1976 to October 21, 1977 and from October 24, 1977 to August 31, 1978. For the first time interval Rubinstein (1981) confirms the findings of MacBeth and Merville (1979) that in-the-money options have relatively high ISDs. However, in the second time interval the original bias observed by Black (1975) reappeared. According to Rubinstein (1985) this reverse was also found by MacBeth (1981) for the year 1978. From these studies Rubinstein (1985, page 478) concludes:

"In total, this evidence is consistent with the hypothesis that striking price (exercise price, author) biases from the Black-Scholes values are statistically significant, the

direction of the bias tends to be the same for most underlying stocks at any point in time, but the direction of bias changes from period to period".

This conclusion is supported by Gemmill (1986), who makes a research for options traded on the London Traded Options Market (LTOM) from May 1978 to July 1983. Gemmill (1986) finds that during this period, out-of-the-money options gave higher ISDs.

Rubinstein's (1985) conclusion is not completely confirmed by Kemna (1988, page 104-111). For the earlier mentioned sample of EOE-options she finds that the exercise price bias is different between the various stocks. For 9 stocks she finds relative low ISDs for at-the-money options, while for the other 5 options she finds relative low ISDs for in-the-money options.

6. Alternatives for the Black/Scholes model.

6.1. Introduction.

In section 4.2 we have already mentioned the fact that a number of modifications of the B/S-model have been developed in order to relax one or more of the assumptions underlying this model. Also a number of tests have been carried out in order to test the performance of the B/S-model and the subsequent developed models for the valuation of call-options. From a review of tests on option pricing models, Galai (1983, page 68) concludes:

"The Black-Scholes performs relatively well, especially for at-the-money options. Deviations from model prices are consistently observed for deep-in and deep-out-of-the money options".

He also concludes:

"No alternative model consistently offers better predictions of market prices than the B-S model. There is some evidence to prefer the constant elasticity of variance model, but it is not conclusive"²¹.

In a more recent overview Hull (1989, page 318) also argues that no model consistently offers better predictions of call-option prices than the B/S-model does.

However, these tests all concentrated on the valuation of call-options having maturities not longer than 9 months. We have already mentioned in section 4.2 that warrants generally have maturities of several years. This makes issues such as dividend payments and the non-stationarity of the standard deviation more important. Therefore we will present a number of these alternative models.

In section 8 empirical tests of option pricing models concentrating on the valuation of warrants will be discussed.

6.2. The Merton Proportional Dividend Model.

It has already been mentioned that the original B/S-model assumes no dividend payments on the stock over the life of the option. Merton (1973) has relaxed this assumption for a rather special dividend policy, dividends are paid continuously so that the dividend yield is constant. This dividend yield can be represented as:

$$g = D/S \quad (6)$$

where:

g = continuous dividend yield;
D = dividend payment per sub-period;
S = stock price.

If the B/S-model is corrected for a continuous dividend payment, the following equation results:

$$C = Se^{-g(T-t)}N(d_1') - Xe^{-rf(T-t)}N(d_2') \quad (7)$$

where:

$$d_1' = \frac{\ln(S/X) + (r_f - g + \sigma^2/2)(T-t)}{\sigma\sqrt{(T-t)}};$$

$$d_2' = d_1' - \sigma\sqrt{(T-t)}.$$

This is the solution to the European call-option problem when

the underlying stock pays dividends continuously at the rate g . In the remainder of this paper we will refer to this model as the 'Merton Model'.

6.3. The binomial model.

In this sub-section we will discuss the binomial model, a model developed by Cox, Ross and Rubinstein (1979)²². In this model it is assumed that the stock price follows a multiplicative binomial process over discrete time periods. At the end of each time period the stock price is multiplied by either the factor ' u ' with a certain probability or the factor ' d ' with the complementary probability. Thus if the current stock price is S , the stock price at the end of the first period will be either uS or dS ²³. This results in a 'binomial tree' for the value of the stock price.

The value of the call-option at its expiration date is the maximum of zero or the difference between the then prevailing stock price and the exercise price. Cox, Ross and Rubinstein (1979) show that the call-option price at $t=0$ can be derived by discounting the expected value of the stock price at maturity according to a risk-neutral investor (following from the 'binomial tree') and the exercise price against the risk-free rate of interest R_f .

Cox, Ross and Rubinstein (1979) prove that in case of an infinite number of time periods, the binomial model converges in the B/S-model.

Important advantage of the binomial model over the B/S-model is that it can incorporate the finding of Merton (1973) that just before a dividend payment early exercise of an American call-option may be profitable. This is possible because the model can incorporate discrete dividend payments²⁴. Consider e.g. a two-period setting, where a dividend is paid at $t=1$. In that case at $t=1$, the dividend is subtracted from uS and dS , and the resulting stock price at $t=1$ is multiplied by either u or d to get the stock price at $t=2$. Because of the dividend payment at $t=1$, it may be profitable to exercise the

call-option at $t=0$. This is the case if the model price at $t=0$ is smaller than the value resulting from direct exercise ($S-X$). Generally the binomial model will have more than 2 iterations, e.g. 50 or 75. The possibility of early exercise can be included, by checking at the iterations just before the dividend payments whether it is profitable to exercise early. If this is the case the model value must be replaced by $S-X$.

6.4. The American constant variance model.

The American constant variance model (from now on the CV-model) is a model originally presented by Schwartz (1977) and later adjusted by Trautmann (1986) and Schulz and Trautmann (1989). The principles underlying the CV-model are partly the same as the principles underlying the earlier discussed B/S-model. In both models it is assumed that the stock price follows a constant variance diffusion process of the form:

$$\frac{dS}{S} = \mu dt + \sigma dz \quad (8)$$

where:

S = stock price;

μ = expected rate of return on the stock;

σ = standard deviation (this is assumed to be constant);

dz = Wiener process.

Together with the continuous trading assumptions, the no-arbitrage partial equilibrium conditions derived for this process has led to the parabolic partial differential equation for option valuation subject to some boundary conditions. Black and Scholes (1973) show that if there are no dividend payments during the lifetime of the option, the value of an option must obey the following partial differential equation:

$$0 = \delta C / \delta t + \frac{1}{2} \sigma^2 S^2 (\delta^2 C / \delta S^2) + r_f S (\delta C / \delta S) - r_f C \quad (9)$$

Using the terminal condition for a call-option:

$$C = \text{Max}[0, S-X] \quad (10)$$

Black and Scholes (1973) were able to derive their (closed-form) equation for the valuation of call-options (see equation 2).

Schwartz (1977), Trautmann (1986) and Schulz and Trautmann (1989) extend this approach to the case where the firm pays a known amount of discrete cash dividend to its shareholders at a specified moment in time²⁵. Following Schulz and Trautmann (1989) we assume:

- (a) the company pays cash-dividend D_k to its shareholders at time t_k^P , where $k=1,2,\dots,K$ and K equals the number of dividend payments during the maturity of the option;
- (b) the ex-dividend date corresponding to dividend D is denominated as t_k , which precedes the dividend payment date t_k^P ;
- (c) the capital market is perfect, i.e. there are no market frictions such as taxes.

Schulz and Trautmann (1989) argue that the stock price at an instant before the ex-dividend date can be written as:

$$S(t_k^-) = S(t_k^+) + \alpha_k D_k \quad (11)$$

where:

- $S(t_k^-)$ = value of a share of common stock, the instant before the ex-dividend date t_k ;
- $S(t_k^+)$ = value of a share of common stock, the instant after the ex-dividend date t_k ;
- α_k = the discount factor applicable between the k th ex-dividend date t_k and the dividend payment date t_k^P ;
- D_k = cash-dividend paid at t_k^P .

Because Merton (1973) has shown that early exercise of a call-option is only optimal just before an ex-dividend date, at each ex-dividend date it must be checked whether the option is more worth held or exercised. This results in a boundary condition:

$$C(S, T-t_k^-) = \text{Max}(C(S, T-t_k^+), S-X) \quad (12)$$

where:

$C(S, T-t_k^-)$ = value of a call-option, the instant before the ex-dividend date t_k ;

$C(S, T-t_k^+)$ = value of a call-option, the instant after the ex-dividend date t_k .

Equation (9), subject to (10, 11 and 12) can only be solved for numerically. A finite difference approximation scheme to solve this system is provided by Schwartz (1977, pages 83-87).

6.5. The constant elasticity of variance model.

In sections 4.2 and 6.4 we have seen that the B/S-model assumes that the volatility is constant over time. Cox and Ross (1976) present a model, in which this assumption is replaced by the assumption that the stock price follows a constant elasticity of variance process of the form²⁶:

$$\frac{dS}{S} = \mu dt + \sigma S^{\Psi-1} dz \quad (13)$$

where:

Ψ = the elasticity factor, in this model it is assumed that:
 $0 \leq \Psi < 1$.

From equation (13) it follows that the stock price has a volatility of: $\sigma S^{\Psi-1}$. Because of the assumption that: $0 \leq \Psi < 1$, the volatility decreases as the stock price increases. This inverse relationship can especially be explained by financial leverage arguments. As the stock price falls, the market value of the firm's liabilities will also fall because of an increased perception of bankruptcy. The decrease in the market value of equity will be larger than the decrease in the market value of debt, which produces a rise of the firm's debt-to-equity ratio. This increase in financial leverage causes an increase in the risk of the equity, which leads to a rise in the stock's volatility. According to Beckers (1980) a similar effect can be observed if the firm has almost no

debt. Since every firm faces fixed costs, which have to be met irrespective of its income, a decrease in income will decrease the value of the firm and at the same time increase its riskiness.

It is interesting to notice that the limiting case of the CEV-model is the case where $\Psi=1$, in that case equation (13) reduces to equation (8), the equation of the constant variance model.

The differential equation for the CEV-model can be written as:

$$0 = \delta C / \delta t + \frac{1}{2} \sigma^2 S^{2\Psi} (\delta^2 C / \delta S^2) + r_f S (\delta C / \delta S) - r_f C \quad (14)$$

Of course equation (14) reduces to equation (9) if $\Psi=1$. For a notation of the CEV-model in a functional form we refer to e.g. Cox and Ross (1976) and Jarrow and Rudd (1983, page 155).

In an empirical study Beckers (1980) investigates the relationship between a change in the stock price and the resulting change in the standard deviation. He uses a sample of 47 stocks with each 1253 daily observations. In his study he finds a significant inverse relationship between stock price and standard deviation for 38 stocks. He also finds that an increase in market leverage, significantly affects the risk to the stockholders. However, also a number of other factors affect this relationship.

Beckers (1980) also tests if one parameter value of Ψ is the same for all stocks. He finds that this is not the case. Therefore Beckers (1980, page 664) concludes that:

"while the CEV class may be supported by the data for an individual stock, it is highly unlikely that a single model can be applied uniformly across all stocks".

As a special case of the general CEV-model, Cox and Ross (1976) present the "square root model", this model has a parameter value Ψ of $\frac{1}{2}$. The diffusion process of the square

root model has the following form:

$$\frac{dS}{S} = \mu dt + \sigma S^{-\frac{1}{2}} dz \quad (15)$$

This model is called the square root model because it assumes that the volatility is inversely related to the square root of the stock value. A formula for the square root model can be derived by substituting the value $\frac{1}{2}$ for the factor $\frac{1}{4}$ in the general equation of the CEV-model. We notice that Beckers (1980) presents a simpler approximation to the formula of the square root model.

6.6. The Longstaff model.

Longstaff (1990) presents closed form expressions for both options that can be extended by the option writer and options that can be extended by the option holder.

In section 4.2 we have already mentioned the fact that especially in the United States the right for the company to extend the warrant's maturity is often included in the warrant trustagreement. Such warrants may be valued using the Longstaff (1990) model for options extendible by the option writer.

Longstaff (1990) argues that his model can also be used for warrants that have a 'step up exercise price' (as discussed in section 4.2). The date at which the exercise price automatically changes can be considered as $t=1$. At $t=1$ early exercise can be optimal because the exercise price increases. If the warrant holder does not exercise his warrant at $t=1$, the warrant is automatically extended until $t=2$. Therefore Longstaff (1990) argues that his pricing formula for options extendible by the option holder is useful for those warrant types.

For a presentation of the Longstaff (1990) models in a functional form we refer to Longstaff (1990).

7. The dilution problem.

7.1. Introduction.

The most important difference between a warrant and a call-option is that when a warrant is exercised, new shares are created, and the exercise price paid for them becomes part of the assets of the firm. This is known as the 'dilution-effect'. This effect has extensively been investigated in finance literature. We will begin our discussion of this effect with a paper written by Galai and Schneller (1978).

7.2. The Galai and Schneller approach.

Galai and Schneller (1978) present their solution for the pricing of warrants, in the form of a one-period model, under the following conditions:

- 1) the firm is assumed to be 100% equity financed;
- 2) the firm's investment policy is not affected by its financing decisions, i.e. the proceeds from issuing warrants are immediately distributed as dividends to the old shareholders;
- 3) the firm does not pay end-of-period dividends;
- 4) the warrants may only be exercised as a block.

The following symbols will be used to present Galai and Schneller's derivation:

V = the value of the firm's assets (without warrants) at the end of the time period (this is also the day the warrants mature);
 N = number of shares of common stock outstanding, before the warrants are exercised;
 n = number of shares to be issued if the warrants are exercised;
 q = the dilution factor = n/N ;
 X = exercise price;
 S_w = price per share without warrants;
 S_x = price per share if warrants are exercised;
 W = model price of a warrant.

If the firm had no warrants outstanding, the price per share

at the end of the time-period would be:

$$S_w = V/N \quad (16)$$

With warrants, the end-of-period value of the firm when warrants are exercised, increases with the exercise price received ($= nX = qNx$). The total number of shares outstanding increases with n , which makes the total number of shares outstanding: $N + n (= N(1 + q))$. The price per share becomes:

$$S_x = \frac{V + NqX}{N(1 + q)} = \frac{V/N + qX}{(1 + q)} = \frac{S_w + qX}{(1 + q)} \quad (17)$$

The warrants will be exercised if the value when exercised is greater than the exercise price, thus if:

$$S_x = \frac{S_w + qX}{(1 + q)} > X, \text{ or rewritten if:}$$

$$S_x = \frac{1}{(1 + q)} (S_w - X) > 0 \quad (18)$$

The warrant will be exercised in exactly the same states of nature as a call-option with the same exercise price, written on the stock of the same firm without warrants. The end-of-period payoffs are presented in table 1.

Table 1: End-of-period payoffs.

	Payoff	
	if $S_w \leq X$	if $S_w > X$
Warrant on a firm	0	$\frac{1}{(1 + q)} (S_w - X)$
Call-option on a firm <u>without</u> warrants	0	$(S_w - X)$
Call-option on a firm <u>with</u> warrants	0	$\frac{1}{(1 + q)} (S_w - X)$

The payoffs to the warrant are a constant proportion of the payoffs to a call-option written on the stock of a firm without warrants. The returns on the warrant, therefore, are perfectly correlated with the returns on a call-option written on the stock of the firm without warrants. In order to eliminate arbitrage possibility, it must be that the value of a warrant, is a proportion of the value of a call-option written on the stock of a firm without warrants:

$$W = \frac{1}{(1 + q)} C \quad (19)$$

where:

C = the value of a call-option written on the stock of a firm without warrants.

Notice from table 1 that the value of a warrant is equal to the value of a call-option written on the stock of the same firm with warrants. In the latter case the relevant price of the underlying share of common stock is S_x , the same share-price as in case of the warrant.

Equation (19) can only be applied in case a firm wishes to issue warrants and use the proceeds as a dividend payment to existing shareholders. Just before the warrant-issue, the value of a call-option written on the stock of a firm without warrants (C) can be calculated, after which the outcome must be multiplied by the dilution factor.

7.3. The dilution correction for outstanding warrants.

In the preceding section we concluded that under the previously specified conditions equation (19) can be used to calculate values of warrants newly to be issued. However, the value of an outstanding warrant cannot simply be calculated by multiplying the dilution factor $(1/(1+q))$ by the value of a call-option derived from an option pricing model, such as the B/S-model. One of the parameters of the B/S-formula is the price of the underlying common stock. The stock price of

a firm with warrants already reflects the potential future exercise of the warrants. Therefore the calculated value of the call-option is the value of a call-option on the stock of a firm with warrants. Of course, this value can not be used in equation (19).

The theoretical solution is to calculate the value of a call-option written on the stock of an otherwise identical firm which does not have warrants in its capital structure. Unfortunately, such an identical firm hardly exists in practice. This makes relationship (19) sec, useless for the valuation of outstanding warrants.

The pricing of outstanding warrants has especially received attention in recent studies by Crouhy and Galai (1988) and Schulz and Trautmann (1989). Crouhy and Galai (1988) discuss the effect of the potential dilution in a discrete-time framework using the binomial model (discussed in sub-section 6.3). In a similar approach Schulz and Trautmann (1989) make use of the B/S-model. Because both approaches come to the same results and because a discussion of Crouhy and Galai (1988) involves many technical details regarding the binomial model, we will take as a guideline for our discussion the work of Schulz and Trautmann (1989). Our discussion of Crouhy and Galai (1988) will be restricted to the most important conclusions from this study.

Following a suggestion made by Black and Scholes in 1973, Schulz and Trautmann (1989) value warrants as options on the equity of the firm instead of options on a share of common stock. In their approach they assume that:

- 1) the value of the firm (V) follows a constant variance diffusion process (see section 6.4). That is, the (unobserved) instantaneous standard deviation of the rate of return on the value of the firm's assets (σ_V), from now on the asset standard deviation, is constant;
- 2) the only financing sources are N shares of common stock and n warrants (with a warrant-ratio of 1).
- 3) no dividends are paid;

4) the warrants may only be exercised as a block.
The current value of the firm can be written as:

$$V = NS + nW \quad (20)$$

Using a similar approach as Galai and Schneller (1978), Schulz and Trautmann (1989) demonstrate that the value of a warrant can be written as:

$$W = \frac{1}{1 + q} C_S \quad (21)$$

where:

$$C_S = (V/N)N(d_1^{ST}) - Xe^{-rf(T-t)}N(d_2^{ST})$$

$$d_1^{ST} = \frac{\ln(V/NX) + (r_f + \sigma_V^2/2)(T-t)}{\sigma_V\sqrt{(T-t)}};$$

$$d_2^{ST} = d_1 - \sigma_V\sqrt{(T-t)}.$$

From equation (21) it can be seen that the warrant is valued as a contingent claim on the equity of the firm, not as a contingent claim on the common stock. Schulz and Trautmann (1989) remark that the difference between the value of the firm and the value of the common stock can be interpreted as anticipated equity dilution. This must equal the value of the outstanding warrants. Notice that equation (21) is consistent with equation (19), because if the firm has no warrants outstanding, the factor V/N in equation (21) reduces to S and the standard deviation of the firm's assets (σ_V) equals the standard deviation of the firm's common stock (σ).

Rewriting equation (20), Schulz and Trautmann (1989) come to equation (22):

$$S = V/N - (n/N)W \quad (22)$$

From Jarrow and Rudd (1983, page 110), Schulz and Trautmann (1989) adopt equation (23):

$$\sigma = \sigma_V \epsilon_{SV} \quad (23)$$

where:

$$\epsilon_{SV} = \frac{\delta S}{\delta V} \frac{V}{S}$$

Schulz and Trautmann (1989) argue that equations (22) and (23) can be solved simultaneously for the unknowns V and σ_V , by a numerical routine for each observed S and σ ²⁷. Then, given the solution pair (V, σ_V) an estimate of the warrant value can be computed using equation (21). With regard to this approach two important remarks are made by Schulz and Trautmann (1989).

The first remark concerns the stock's elasticity. From equation (23) it can be concluded that the stock's elasticity is a function of the firm value and time. Therefore the stock standard deviation (σ) changes over the life of the outstanding warrants, even under the earlier introduced assumption that the asset standard deviation (σ_V) is constant. This is also confirmed by Crouhy and Galai (1988). The second remark concerns the finding of Schulz and Trautmann (1989) that the stock standard deviation (σ) is always below the asset standard deviation (σ_V). This is also found by Crouhy and Galai (1988), who explain this phenomenon by arguing that a warrant can be replicated in a dynamic framework by issuing additional shares and investing the proceeds in government bonds²⁸. According to Crouhy and Galai (1988) this tends to reduce the risk of the assets of the firm and, hence of its equity.

Schulz and Trautmann (1989) also investigate the differences between simply calculating warrant prices using the B/S-model (equation (2)) and calculating warrant prices using the precise warrant valuation model (equation (21)). Schulz and Trautmann (1989) assume the parameters S , $T-t$, X , q , σ_V and r_f as given and solve equations (22) and (23) simultaneously for the model input parameters V/N and σ . From this investigation they conclude that, even if an extremely high

dilution factor is assumed, the bias resulting from simply using the B/S-model is very small. Substantial differences only exist for deep out-of-the-money warrants and out-of-the-money near maturity warrants. Therefore Schulz and Trautmann (1989, page 16) conclude:

"To obtain warrant values with acceptable accuracy, adjustments to the Black/Scholes formula are not needed except perhaps for deep out-of-the-money warrants".

7.4. Block exercise versus sequential exercise.

One problem with the approaches described in sections 7.2 and 7.3 is that warrants are to be exercised in one large block. This is not necessary the case for call-options of the American type. A large body of literature has appeared on this subject²⁹. We will shortly review the most important elements from these studies. Important difference is whether warrants are held by a single profit-maximizing monopolist or by single competing individuals. First the situation of the monopolist will be discussed.

i) The situation for the single profit-maximizing monopolist.

Emanuel (1983) argues that a single profit-maximizing monopolist owning all warrants (from now on: a monopolist) may generally prefer to exercise his warrants sequentially instead of in a large block³⁰. The advantage of sequential exercise may arise because if the monopolist exercises part of his warrants this leads to a creation of new shares and may therefore lead to changes in the dividend policy and capital structure of the firm.

Assuming a company financed with only common stock and warrants, Emanuel (1983) presents an example in which such a company pays dividends at a constant rate of the assets. If warrants are exercised the dividend rate per share declines. This hurts all the shareholders including the monopolist who exercised part of his rights. However Emanuel (1983) states

that since the aggregate value of all claims on the firm equals the value of the firm and warrant exercise has not made the firm more or less valuable, the class of warrant holders must have gained from the change, because the class of shareholders has lost. Only if the company follows a so-called "neutral dividend policy", as specified by Emanuel (1983), the existing shareholders may avoid the disadvantage that arises from the sequential exercise.

Emanuel (1983) argues that the profits to be made by a monopolist from sequential exercise will only be substantial when the potential dilution effects are extreme. This is an important problem if Emanuel's (1983) theory is to be tested.

Spatt and Sterbenz (1988) discuss two other possibilities other than a change in the dividend policy that cause sequential exercise to be beneficial for a monopolist.

The first possibility is that the warrant exercise proceeds are used to repurchase shares. Despite the fact that the warrant holder forfeits the premium above parity on those warrants exercised, a net benefit to the warrant holder may result, because the change in capitalization causes a spread out of the stock price, and thus an increase in the volatility. This is beneficial for the warrant (holder) because of the positive relation between the volatility and the warrant price.

The second possibility is that the funds are used to expand the firm's investment project. This increases the riskiness of the firm and therefore leads to an increase in the volatility. The benefits are the same as under the first possibility. Spatt and Sterbenz (1988) notice that the second possibility was already recognized by Cox and Rubinstein (1985, pages 397-399) who present an example in which the increase in value of the unexercised warrants exceeds the forfeiture on the premium above parity for the exercised warrants³¹.

Spatt and Sterbenz (1988) also mention three possibilities

available to the existing shareholders in order to avoid disadvantages from sequential exercise³².

The first possibility to neutralize disadvantages from sequential exercise strategies is to issue new warrants with the same exercise price and maturity as the warrants just exercised and to use the proceeds of the exercise and the new issue to repurchase the corresponding number of equity shares³³. In this case both the wealth of the firm and its capital structure remain unchanged. In fact it is much like the "neutral dividend policy" described by Emanuel (1983).

The second possibility is to invest the proceeds from any warrant exercise in a riskless zero-coupon bond that matures at the warrant's expiration. This causes holding warrants until maturity (and therefore block exercise) to be the optimal strategy for warrant holders.

The third possibility for the firm is not to pay a regular periodic dividend but to pay an extraordinary dividend to shareholders with the proceeds of any warrant exercise. Spatt and Sterbenz (1988, page 494) argue:

"Although warrants are subject to some antidilution protection in practice, it appears that the firm enjoys considerable latitude".

This reinvestment policy makes it not beneficial for warrant holders to follow the sequential exercise policy.

We complete this discussion by remarking that possible benefits for monopolistic warrant holders depend upon the reinvestment policy followed by the firm. It seems reasonable to us to assume that a company (read: the existing shareholders) generally will choose for a policy that makes it (practically) impossible for warrant holders to derive any advantage from sequential exercise.

ii) The situation for competitive warrant holders.

Constantinides (1984) discusses the case where warrants are held in small amounts by a number of individuals. These

individuals have the possibility to exercise all or part of their warrants at each time that the warrants are exercisable. Important assumption is that the warrant holders act as competing individuals, who can not form binding arrangements.

Just as in the case of the monopolist, exercise of part of the warrants leads to a creation of new shares and may therefore lead to changes in the dividend- and reinvestment policy of the firm. If this is the case, the action of one warrant holder influences the value of a warrant held by someone else.

Constantinides (1984) proves that there exist multiple competitive equilibria. For at least one equilibrium, the price of a warrant equals the price that is established in the block exercise equilibrium. In the other equilibria the price of a warrant is less than the price established in the block equilibrium.

Using a model developed by Cox and Rubinstein (1985, pages 396-399) we will illustrate the nature of these equilibria. First the case will be presented where the value of the warrants held by competing individuals, having the possibility to exercise their warrants sequentially, is the same as the value the warrants would have if they were held by a monopolist, constrained to exercise all of his warrants in a large block.

Example 2:

In this example we use a special version of the binomial model. This model, developed by Cox and Rubinstein (1985, page 396-399), is described in more detail in appendix B. In this model a two-period discrete time process is assumed, with one period remaining in the life of the warrants. The firm has N shares of common stock and two warrants, warrant X and warrant Y , outstanding. Any warrant holder may, if he wishes, exercise at $t=0$. If he does so, he receives one newly issued share of common stock in return for the exercise price X . This exercise price becomes immediately part of the firm's assets. Then one period passes, and the value of the assets in the firm is multiplied by either u or d . At $t=1$ the warrant holders again have the possibility to exercise or else let the warrants expire, since this is now the expiration date. Thus each warrant holder has the choice between exercising at $t=0$ or waiting until $t=1$. The value of

each strategy depends on the strategy followed for the other warrant.

In an example Cox and Rubinstein (1985) present the following estimates of the parameters:

$V = f\ 1.000.--$; $N = 4$; $X = f\ 26.--$; $u = 1.8$; $d = 0.1$; $R_f = 5\%$. In table 2 the resulting situation is presented.

Table 2.

	Both warrants are exercised at $t=0$	Only warrant X is exer- cised at $t=0$	None of the warrants are exercised at $t=0$
	(1)	(2)	(3)
Value of war- rant $X = W_X$	$f\ 149.33$	$f\ 148.74$	$f\ 150.44$
Value of war- rant $Y = W_Y$	$f\ 149.33$	$f\ 152.28$	$f\ 150.44$
$W_X + W_Y$	$f\ 298.66$	$f\ 301.02$	$f\ 300.88$

If the warrants were held by a monopolist, he would exercise warrant X at $t=0$ and hold warrant Y. This would result in a total warrant value of $f\ 301.02$. If the warrants were held by a monopolist under the constraint of block exercise, he would hold his warrants at $t=0$, which would result in a total warrant value of $f\ 300.88$. The value of $f\ 300.88$ would also result if the warrants were held by a competing individuals, having the possibility of sequential exercise, because neither warrant holder would have an incentive to exercise his warrants at $t=0$. Therefore the warrant value of competing individuals is less than the warrant value of a monopolist, having the possibility of sequential exercise, but equal to the value of a monopolist under the block constraint.

Constantinides (1984, page 382) argues that also competitive equilibria exist where the value of the warrants under sequential exercise is less than the value of the warrants under block exercise. In an example he describes the situation that the firm "threatens" to pay out a liquidating dividend if a certain number of warrants is exercised. We will illustrate this point in example 3.

Example 3:

Using the same data as in example 2, we assume that the firm pays out a liquidating dividend if at least one warrant is exercised at $t=0$. This liquidating dividend will be paid immediately after $t=0$.

If only one warrant is exercised at $t=0$, the value of this warrant is the liquidating dividend minus the exercise price paid, that is:

$$\frac{V + X}{N + 1} - X = \frac{1000 + 26}{4 + 1} - 26 = f 179.20$$

The unexercised warrant has no value in this case.
If both warrants are exercised, the value per warrant is:

$$\frac{V + 2X}{N + 2} - X = \frac{1000 + 52}{4 + 2} - 26 = f 149.33$$

This leads to the situation presented in table 3.

Table 3.

	Both warrants are exercised at $t=0$	Only warrant X is exer- cised at $t=0$	None of the warrants are exercised at $t=0$
	(1)	(2)	(3)
Value of war- rant X = W_X	f 149.33	f 179.20	f 150.44
Value of war- rant Y = W_Y	f 149.33	f 0	f 150.44
$W_X + W_Y$	f 298.66	f 179.20	f 300.88

In table 3 we see that the monopolist warrant holder chooses not to exercise his warrants at $t=0$ and to wait until $t=1$. This results in a total warrant value of f 300.88. The situation would be different if the warrants were held by single competing individuals. The holder of warrant X would expect the holder of warrant Y to exercise "too early", leaving him with a worthless warrant. The same reasoning is followed by the holder of warrant Y. Therefore both warrants would be exercised at $t=0$. The resulting warrant value of f 298.66 is less than the warrant value that would be realized in case of block exercise. In fact a "prisoners dilemma" exists.

According to Constantinides (1984) and Cox and Rubinstein (1985, page 399) the situation that competitive shareholders, having the possibility to exercise their warrant sequentially, are worse off than a monopolist under the constraint of block exercise, only occurs if extreme assumptions are made (as is the case in example 3). In addition we notice that the original Cox and Rubinstein

(1985, page 396-399) model, which does not include dividend payments, never leads to a solution where competitive warrant holders are worse off under the possibility of sequential exercise³⁴.

iii) Conclusion.

Summarizing we argue that the possibility of sequential exercise may lead to an advantage for monopolists, but that according to Spatt and Sterbenz (1988) the existing shareholders have several possibilities to neutralize these advantages if the proceeds from the warrant exercise are used in "the right way".

On the other hand the possibility of sequential exercise may have a negative effect for competitive warrant holders, but according to Constantinides (1984) and Cox and Rubinstein (1985) such a situation only occurs under extreme conditions³⁵.

Therefore in a recent paper on this subject Spatt and Sterbenz (1988, pages 494-495) argue:

"Our analysis of the obstacles to sequential exercise helps to justify the frequent simplifying restriction that warrants or convertible securities are valued as if exercised as a block and (...) to sustain much of the long-standing theory of the valuation of warrants".

This brings us back to the theory presented in sections 7.2 and 7.3.

8. Tests of option pricing models for the valuation of warrants.

8.1. Introduction.

In this section empirical research upon the valuation of warrants will be discussed. In advance we notice that all of the empirical studies considered in this section, have abstracted from the possibility of sequential exercise and

implicitly or explicitly assumed that warrants will be exercised in one large block. This means that the issues raised in section 7.4 have been abstracted from.

8.2. Noreen.

Noreen (1982) tests a sample of US warrants outstanding on April 18, 1975. This sample is drawn from the 245 warrants outstanding in the United States on that day. From this sample 195 warrants were eliminated mainly due to one or more of the following reasons: (1) they were issued by a real estate investment trust (REIT); these warrants were eliminated because the dividend yield of a REIT is difficult to estimate; (2) the warrant had complicating exercise provisions (see section 4.2); (3) not enough data were available for the specific warrant.

The models tested were the Merton model and the ad hoc models of Kassouf (1969) and Shelton (1967b)³⁶, from now on the Kassouf model, respectively the Shelton model. Noreen (1982) corrects the outcome from the Merton model by the dilution factor $(1/(1+q))$.

Because the ad hoc models require cross-sectional estimation of parameters (see section 3) the sample of 50 warrants was randomly split in a Calibration Subsample and a Testing Subsample, each consisting of 25 warrants. The parameters of the ad hoc models were estimated using the Calibration subsample. We notice that the standard deviation for the Merton model was estimated using monthly historical stock price data for the 48 months up to April 1975.

Noreen (1982) calculates prediction errors for the three models, defined by the ratio:

$$\text{prediction error} = \frac{\text{model price} - \text{market price}}{\text{market price}}$$

The prediction errors for the three models are presented in table 4.

Table 4: Mean prediction errors and mean absolute prediction errors for the models tested by Noreen (1982).

	Mean absolute prediction error	Mean prediction error
Merton model	59 %	- 48.8%
Kassouf model	59.2%	29.6%
Shelton model	139.3%	117.9%

Based on the results from table 4, Noreen (1982) concludes that the Kassouf model performs "best".

However, Galai (1989) notes that Noreen (1982) erroneously introduces the dilution factor $(1/(1+q))$ for the Merton model. In section 7.3 we have seen that a dilution correction is not necessary for outstanding warrants. This explains (part of) the negative mean prediction error for the Merton model. Unfortunately Noreen (1982) does not provide any information with regard to the dilution factor, which makes it impossible to see which results would occur if this factor would be omitted.

8.3. Noreen and Wolfson.

Noreen and Wolfson (1981) are interested in the valuation of executive stock-options. Because executive stock-options are not traded, they select a sample of US warrants that have characteristics resembling those of executive stock-options. Their sample consists of 52 observations of 52 different warrants. The observations are made in the period from April 1969 until December 1978. The most important characteristics of the warrants selected are: (1) the warrants are not issued by a REIT; (2) none of the warrants has complicated exercise provisions; (3) enough data are available, concerning all relevant parameters; (4) the remaining time to maturity of the warrants is between 2 and 6 years and (5) the warrants are at-the-money³⁷. Notice that conditions (1), (2) and (3) are the same as in Noreen (1982). Conditions (4) and (5) are

added in order to make the warrants comparable to executive stock-options. The models tested are the Merton model and the square root model. The square root model (described in section 6.5) is corrected for continuous dividend payments³⁸, therefore this version of the square root model will be referred to as the 'altered square root model'.

Even as Noreen (1982), Noreen and Wolfson (1981) erroneously correct the outcomes for both models by the dilution factor $(1/(1+q))$. Furthermore we notice that the standard deviation for both models is estimated using weekly stock price returns over the 50-week period immediately preceding the maturity date.

Noreen and Wolfson (1981) conclude that the estimated warrant prices from the two models are never more than 5 percent apart for the whole sample. Both models produce a mean absolute prediction error of 16.8%. The mean prediction error is -5.4% for the Merton model and -5.2% for the altered square root model. It is interesting to notice that the prediction errors for the Merton model are much smaller than in Noreen (1982). This might be due to one or both of the following factors:

- 1) in this sample only at-the-money call-options are included; in section 6.1. we have already seen that the B/S-model performs best for at-the-money options;
- 2) the estimation of the standard deviation by Noreen and Wolfson (1981) is more precise than Noreen's (1982) estimation: instead of monthly returns, weekly returns have been taken and the standard deviation is estimated over the last 50 weeks instead of the last 48 months.

Of course, also other factors may be responsible for this lower prediction error³⁹. A factor that is probably of low importance is the fact that Noreen and Wolfson (1981) only include warrants with maturities up to 6 years. This factor is probably not relevant because Noreen's (1982) sample only included one warrant with a maturity longer than 6 years.

We have already mentioned the fact that Noreen and Wolfson

(1981) erroneously correct for the dilution factor $(1/(1+q))$. Fortunately, they also present the results in case the dilution factor is omitted. In that case the mean absolute prediction error becomes 16.1% for the Merton model (instead of 16.8%) and 16.2% for the altered square root model (instead of 16.8%). The mean prediction error changes from low negative to low positive for both models: +3.4% for the Merton model (instead of -5.4%) and +3.6% for the altered square root model (instead of -5.2%).

8.4. Folks and Ferri.

Folks and Ferri (1987) study the pricing of 10 warrants attached to Eurobonds, in short to be referred to as Eurowarrants. Folks and Ferri (1987) argue that the limited size of the issues of Eurobond-warrants make for a very thin market. Therefore they argue that the only time for which a reasonable reliable market price is obtainable, is the date of issue. The procedure Folks and Ferri (1987) use to determine the 'market value of the warrants' is to take the issue-price of the bond-warrant package and subtract the value of the 'naked bonds' from the issue-price. The value of the bonds can be determined by using the following equation:

$$BV = \sum_{t=1}^m \frac{I}{(1 + k_b)^t} + \frac{F}{(1 + k_b)^m} \quad (24)$$

where:

BV = the value of the 'naked bond';

m = the maturity of the bond;

I = interest paid each year;

k_b = cost of debt with similar risk and maturity;

F = bond's redemption value at maturity.

Folks and Ferri estimate the factor k_b in equation (24) as the average yield on the month of issue on all Eurobond straight debt which was denominated in the same currency.

Two models are tested: the B/S-model and the Merton model. The outcomes for these models are multiplied by the dilution

factor $(1/(1+q))$. Notice that in section 7.2 we have concluded that the dilution factor may only be used in case (1) the warrants are newly to be issued and (2) the proceeds of the warrant-issue are to be distributed as a dividend payment amongst existing shareholders. In this case the first condition is fulfilled but Folks and Ferri (1987) do not indicate whether this is also the case for the second condition. Therefore we question whether the dilution factor used by Folks and Ferri (1987) is correct.

The standard deviation is estimated by using monthly returns of the underlying stock for two years preceding the issue.

Folks and Ferri (1987) present a table in which the 'market prices' and model prices are compared. In order to be able to compare their results with the results derived by Noreen (1982) and Noreen and Wolfson (1981) we have calculated mean prediction errors and mean absolute prediction errors for both models. The mean absolute prediction error for the B/S-model is 103.12%, for the Merton model it is 35.3%. The mean prediction error is 103.12% for the B/S-model and -15.9% for the Merton model.

Caution should be exercised in interpreting the results of this study, because the 'market price' of the warrants is calculated as the difference between the issue-price and the value of the naked bonds. The issue-price may have been fixed by the issuing company in such a way that it would have been higher or lower than the market value of the warrant-bond package. This may have been based on rational grounds, but may also have been based on an erroneous estimation of the issuing company. Besides that, the calculation of the bond-value may have been biased, due to an erroneous estimation of k_b . This would lead to a biased estimate of the warrant's 'market value'.

The studies considered until now all have in common that relatively little observations are used, differing from 10 (Folks and Ferri (1987)) to 52 (Noreen and Wolfson (1981)). The studies to be discussed in the remainder of this section

all use a (much) larger number of observations.

8.5. Stucki and Wasserfallen.

Stucki and Wasserfallen (1989) test a sample of 44 Swiss warrants, using weekly data for the period between January 8, 1986 and February 25, 1987. The following option pricing models are tested:

- 1) the B/S-model;
- 2) the Merton model;
- 3) the B/S-model, using an ad hoc dividend correction as suggested by Black (1975); this model will from now on be referred to as the 'corrected B/S-model';
- 4) the binomial model, using a correction for discrete dividends (as described in section 6.3).

No dilution correction is made. Two different estimates are used for the standard deviation:

- 1) the historical standard deviation (HSD) based on the most recent 52 weekly observations of the return on the underlying common stock;
- 2) the implied standard deviation (ISD) of the same warrant, calculated for the previous week using the same model; for computational considerations this approach has not been used for the corrected B/S-model.

Just as in the studies discussed earlier in this section, errors are calculated. Unfortunately Stucki and Wasserfallen (1989) do not use the same ratio as defined by Noreen (1982). Therefore their ratio will be referred to as the 'estimation error', defined as:

$$\text{estimation error} = \frac{\text{model price} - \text{market price}}{\text{model price}}$$

The mean estimation error for the four models and two volatility estimations are presented in table 5.

In the discussion of the results presented in table 5 we will first pay attention to the estimation errors in case the HSD

is calculated.

Table 5: Mean estimation errors for the models tested by Stucki and Wasserfallen (1989).

	HSD	ISD for the previous week
B/S-model	- 4.81%	-0.28%
Merton model	-22.61%	-0.18%
corrected B/S-model	-22.18%	not calculated
binomial model	-21.65%	-0.21%

Notice first that in case of the HSD, all models undervalue warrant prices. The negative estimation error for the Merton model is particularly interesting because in section 8.3 we saw that Noreen and Wolfson (1981), after omitting the dilution factor, conclude that the Merton model tends to overvalue warrant prices. Another interesting fact from table 5 is that even the B/S-model undervalues warrant prices. Because of the fact that this model does not take dividend payments into account it would be expected to overvalue warrants rather than to undervalue them (see also the results derived by Folks and Ferri (1987)). Stucki and Wasserfallen (1989) do not give an explanation for this phenomenon.

Stucki and Wasserfallen (1989) notice that in case the ISD of the previous week is used, the estimation errors decline relative to the case where the HSD is used. We do not find this result surprising. Stucki and Wasserfallen (1989) in fact simply compare the ISD of week_{t-1} with the ISD of week_t . However, we do question the usefulness of this approach. If the question is to be asked whether the ISD derived from warrant prices is a better predictor of the future standard deviation than the HSD is, a test as described in scheme 1 would be preferred over the test carried out by Stucki and Wasserfallen (1989). If on the other hand, the option pricing model is tested for the valuation of warrants, the input of the ISD of week_{t-1} is not

very useful. A possible erroneous specification of the model would result in a biased ISD for week_{t-1} . This makes a comparison between the two columns of table 5 meaningless.

8.6. Schulz and Trautmann.

Trautmann (1986) tests the American constant variance model (see section 6.4) for a sample of German warrants during the period of January 1, 1979 until June 30, 1985. Because this valuation model, amongst other models, is also included in the study of Schulz and Trautmann (1989), even for a larger time period and a more extensive sample, we will leave the empirical results of Trautmann (1986) undiscussed and continue our discussion with the paper written by Schulz and Trautmann (1989).

Schulz and Trautmann (1989) test a sample of 46 German warrants, using weekly data for the period between January 1, 1979 and December 30, 1986. These warrants were drawn from a larger sample of which warrants were eliminated because they had: (1) a market that was not liquid; (2) not enough data available concerning all relevant parameters; (3) an exercise price denominated in a foreign currency.

The following option pricing models were tested:

- 1) the B/S-model applied to the stock price net of the present value of the escrowed dividends, from now on to be referred to as the 'adjusted B/S-model';
- 2) the American constant variance model (see section 6.4), from now on to be referred to as the 'American CV-model';
- 3) the constant elasticity of variance model (see section 6.5), from now on to be referred to as the CEV-model.

Schulz and Trautmann (1989) do not make a correction for the dilution effect. With regard to the stock price they notice that a correction is made in case the warrant-conditions include the provision that shares issued upon warrant exercise first entitle the new shareholder to dividends over the business year in which they are exercised⁴⁰.

The standard deviation is estimated using weekly log returns over the 52-week period immediately preceding the measurement date. In case of the CEV-model a so-called "maximum likelihood procedure" is followed for the simultaneous estimation of the volatility parameter and the elasticity factor⁴¹. Contrary to the assumptions underlying the CEV-model (see section 6.5) Schulz and Trautmann (1989) do not assume a pre-specified range for the elasticity factor (Ψ).

The first conclusion drawn by Schulz and Trautmann (1989) is that the elasticity parameter of the CEV-model is extremely instationary. In some years a direct relation between the volatility and the stock price was dominant for all stocks ($\Psi > 1$) and in other years an inverse relation was predominant ($\Psi < 1$). Therefore Schulz and Trautmann (1989, page 21) conclude:

"The unpredictability of this instationarity gives rise to doubts on the superiority of the CEV models compared to the simple CV model".

Schulz and Trautmann (1989) calculate prediction errors (as defined by Noreen (1982)) for all three models. These prediction errors are presented in table 6.

Table 6: Mean prediction errors and mean absolute prediction errors for the models tested by Schulz and Trautmann (1989).

	Mean absolute prediction error	Mean prediction error
Adjusted B/S-model	20.5%	- 5.2%
American CV-model	19.6%	- 0.1%
CEV-model	19.8%	1.4%

Schulz and Trautmann (1989) argue that the negative mean prediction error resulting from the adjusted B/S-model is due to the fact that this model does not take early exercise resulting from dividend payments into account. Schulz and

Trautmann (1989) argue that, notwithstanding the relative small mean prediction errors and mean absolute prediction errors, substantial differences exist for specific warrant issues, even on an average basis.

Schulz and Trautmann (1989) also carry out regressions for the American CV-model and the CEV-model in order to examine the differences between market prices and model values. The following regression results were found:

- price order?*
- 1) both models tend to undervalue low-priced warrants and to overvalue high-priced warrants;
 - 2) there exists a significantly positive relationship between relative prediction errors and time to expiration;
 - 3) both models appear to overprice warrants on stocks with high potential dilution factors and to underprice warrants on stocks with low potential dilution.

In light of the results derived in section 7 especially the last result seems interesting. As explanations for this result Schulz and Trautmann (1989) mention:

- 1) the possibility that it is due to a correlation, because high dilution factors were especially observed in the "overpricing period";
- 2) the possibility that potential dilution is not completely anticipated in the observed stock prices.

The conclusion for the Schulz and Trautmann (1989) paper is that the American CV-model and the CEV-model work equally well and that their pricing performance is similar to that reported for stock options.

8.7. Lauterbach and Schultz.

Lauterbach and Schultz (1990) test a sample of 39 US warrants, using daily data for the period of January 1971, through December 1980. Their sample of 39 warrants is drawn from a sample of 100 warrants. From this sample 61 warrants were eliminated because: (1) the warrants had complex exercise provisions; (2) not enough data were available; (3) the warrants were written on a company, named as a take-over

candidate by the Wall Street Journal, these warrants were eliminated because of the uncertain treatment of warrants in the event of a merger. In addition individual warrant observations were eliminated because (1) warrant prices were less than 1\$; (2) arbitrage conditions were violated; (3) the maturity was extended in the period 3 months after the observation; (4) fewer than 20 observations in a quarter were available.

Lauterbach and Schultz (1990) first test a version of the B/S-model that is corrected for dividends and dilution. The dilution correction is in line with equation (21) presented by Schulz and Trautmann (1989). In this equation the warrant is valued as an option on the equity of the firm instead of an option on the common stock. Lauterbach and Schultz (1990) adjust for dividends by subtracting the present value of the dividends from the equity value. This adjusted version of the B/S-model will from now on be referred to as the 'alternative B/S-model'. This alternative B/S-model is used to calculate ISDs.

Lauterbach and Schultz (1990) first calculate an average of the daily ISDs for each warrant during each quarter. The average ISDs are then compared to the actual (realized) standard deviations over the following quarter (see the procedure of scheme 1). Notice that the actual standard deviation is the standard deviation of the equity (defined as: $S + (n/N)W$). Therefore this standard deviation is like σ_v defined by Schulz and Trautmann (1989), except for the fact that Schulz and Trautmann (1989) assume that no debt is outstanding. Lauterbach and Schultz (1990) conclude that the average ISD over the warrant quarters is 55.6%. The average equity standard deviation realized the following quarter is a similar 59.4%. Lauterbach and Schultz (1990) also note that the average realized stock standard deviation (σ) in the subsequent quarter is 41.5%. This result is in accordance with Crouhy and Galai (1988) and Schulz and Trautmann (1989), who also come to the conclusion that equity

volatility is larger than stock volatility.

Following a procedure suggested by Rubinstein (1985), Lauterbach and Schultz (1990) test the alternative B/S-model by running a regression for each warrant each quarter of the testing period, in which the ISD on day_t is regressed against (1) the default free interest rate of the previous day and (2) the percentage that the warrant is in- or out-of-the-money on the previous day. From this regression they conclude that (1) little evidence exists of a relation between model price errors and interest rates and (2) the ISDs of warrants are inversely related to the value of the underlying equity. This second result led Lauterbach and Schultz (1990) to conclude that models allowing for an inverse relation between equity value and equity volatility such as the CEV-model are a promising substitute to the alternative B/S-model⁴².

The next test of Lauterbach and Schultz (1990) is a test between the pricing performance of the alternative B/S-model and an adjusted version of the square root model. The version of the square root model tested is the simplified version of this model, presented by Beckers (1980). This model is corrected for dividends and dilution in the same way as the B/S-model. Therefore this model will be referred to as the 'alternative square root model' (alternative SR-model). To compare the alternative B/S-model and the alternative SR-model, ISDs are estimated for each warrant each day using both models. Daily observations are then weighted by the derivative of the warrant price with respect to the standard deviation and averaged over a quarter to get ISDs for each warrant each quarter. Alternative SR- and alternative B/S-model ISDs are then used to price the warrants in the subsequent quarter. Tests of these model values show that the alternative SR-model is a consistently more accurate predictor of market prices than the alternative B/S-model. Tests of Lauterbach and Schultz (1990) also show that the

difference in accuracy increases for longer maturities. Therefore Lauterbach and Schultz (1990, page 1207) conclude: "Besides indicating that the SRCEV model (alternative SR-model, author) is particularly advantageous for pricing long-lived warrants, these results could be interpreted as indirect evidence that CEV-models may be more important for pricing warrants than shorter-lived options".

9. A comparison of implied standard deviations derived from warrant prices with implied standard deviations derived from warrant-like instruments.

9.1. The comparison of implied standard deviations derived from warrant-prices with implied standard deviations derived from call-option prices.

In this paper we have reviewed both the theoretical and the empirical analysis that has been made on the valuation of warrants. A possible method for investigating the pricing of Dutch warrants is the calculation of model prices using one of the option pricing models presented in section 6, and to calculate prediction errors for these warrants. The results from such a study can be compared with the results of the pricing of US-, Swiss- and German warrants as has been investigated by respectively Noreen and Wolfson (1981), Stucki and Wasserfallen (1989) and Schulz and Trautmann (1989).

However, because of the fact that in the Netherlands also long term call-options, having maturities resembling those of warrants, have been issued, another possibility of studying warrant prices is the comparison of warrant prices with the prices of long term call-options. Such a comparison is possible by calculating implied standard deviations (ISDs) for warrants and call-options ⁽⁴³⁾. The ISD derived from a call-option (warrant-) price can be considered as representing the call-option (warrant-) price. Because

Merton (1973) has shown that a positive relation exists between the option price and the standard deviation, a relatively higher ISD means that the call-option (warrant) is valued relatively higher. In case a call-option and a warrant, written on the same stock, are identical, the same ISD is expected. In comparing differences between ISDs derived from call-option and warrant-prices the following factors should be taken into account:

- 1) in section 4.2. we have already mentioned the fact that warrants may have special conditions, e.g. they may be callable; of course these special conditions influence the price of a warrant;
- 2) the model with which ISDs are calculated may have been wrongly specified, this may result in e.g. a 'time to maturity bias' and/or an 'exercise price bias' (see sub-sections 5.2.3. and 5.2.4.), therefore the use of several option pricing models should be considered.

We also mention two factors that are different for warrants and call-options, but which we believe will not cause large price differences between warrants and call-options:

- 1) the use of anti-dilution clauses is different for warrants and call-options; in appendix C the most important information on this subject is summarized; from this appendix we conclude that although the application of anti-dilution clauses differs, on balance the differences tend to outweigh each other: in some cases warrant holders are better off, in other cases holders of call-options are better off;
- 2) in section 4.2 we have seen that transaction costs (to be paid by private persons) are somewhat different for warrants than for call-options; these differences decline if large numbers of shares of common stock are to be bought upon exercise (i.e. ≥ 500); therefore we remark that this difference in transaction costs may not be responsible for large price differences between warrants and call-options.

We notice that in case possible differences between ISDs

derived from warrant- and call-option prices can not be explained by one of the factors described above, factors additional to those described in this paper must be responsible for differences in the valuation of call-options and warrants. It should be emphasized that the dilution-effect may not cause ISDs to be different. In section 7.2 (table 1) we have seen that the dilution effect does not differ for a warrant and a call-option written on a firm with warrants. Besides that the dilution effect will be anticipated in the stock price (see section 7.3).

For an explicit discussion of the comparison of implied standard deviations derived from warrant prices with implied standard deviations derived from call-option prices we refer to Veld and Verboven (1991b).

9.2. The calculation of implied standard deviations from conversion right prices.

Warrants are often issued in combination with bonds. A finance instrument that is close to a warrant-bond package is the convertible bond. The conversion right attached to a convertible bond is generally defined as a warrant that can only be exercised if the accompanying bond is used as a payment of the exercise price⁴⁴. In case a conversion right may only be exercised at its expiration date by redeeming the accompanying bond at its par value, the conversion right is identical to a European warrant. In that case a comparison of the ISDs derived from the conversion right and the warrant (as described in section 9.1) is a way to compare the pricing of warrants and conversion rights. However, the conditions of convertible bonds are generally (much) more complicated than in the simple case described above. In this sub-section we will review these complications and, whenever available, present a method to overcome the problems⁴⁵. The following problems may occur in the calculation of ISDs from conversion rights:

- 1) the bond and the conversion right are not separately

- tradeable;
- 2) the convertible bond contains a sinking fund provision;
 - 3) the convertible bond can be converted before its expiration date;
 - 4) the convertible bond, denominated in another currency than the underlying shares of common stock, is convertible at a fixed exchange rate.
 - 5) the convertible bond is callable;

In the remainder of this sub-section these problems will be discussed more extensively.

ad. 1) Bond and conversion right are not separately tradeable.

The first problem that may occur is that the bond and conversion right are not separately tradeable. In that case it is necessary to calculate the value of the conversion right. The market value of the convertible bond has two components: (1) the bond value and (2) the value of the conversion right. A possible procedure is to calculate the bond's value using equation (24) and to subtract this from the market value. The cost of debt (k_p) can be estimated using the effective return of a bond, having a similar maturity, issued by the same company. In case the company has only a bond with a different maturity outstanding, the effective rate of return may e.g. be calculated using interpolation⁴⁶.

Additional problems may be: (1) the market value of the convertible bond is presented net of the current interest and (2) in the conditions of convertible bonds a statement is included that over the year in which the bond is converted, the holder of the convertible bond is not entitled to an interest payment, because he will receive dividends on the underlying stock over that year (this dividend is to be paid in the next year).

ad. 2) The inclusion of a sinking fund provision.

Convertible bonds may contain a sinking fund provision in

which case the convertible bond includes a series of conversion rights, portion of which expire on each sinking fund date. This causes a problem if ISDs are to be calculated. We will illustrate this in example 4, which is partly derived from Veld and Verboven (1991a).

Example 4:

A company issues convertible bonds on January 1, 1990. The bonds are issued at their par value of f 1000.-- and may be converted into 8 shares of common stock at their expiration date (therefore the exercise price per conversion right is f $1000/8 = f 125.--$). The bonds will be redeemed in five equal parts, for the first time on January 1, 1991 and for the last time on January 1, 1995. Because the holder of the convertible bond does not know in which series his bond will be redeemed, the remaining time to maturity is unknown. Veld and Verboven (1991a) suggest two approaches: (a) the 'average maturity' (AM) approach and (b) the 'part warrant' (PW) approach. These approaches will be explained below:

- ad. a) The AM-approach: the conversion right can be considered as a warrant, with a maturity equal to the average maturity of the convertible bond;
- ad. b) The PW-approach: the conversion right can be considered as a portfolio of part-warrants; the maturity of each part-warrant is one year longer than the maturity of the preceding part-warrant; the portfolio-weights are equal to the possibility of redemption in the specific year.

In this example we will demonstrate the calculation of ISDs for the B/S-model using both approaches. Besides the maturity the following parameters are relevant:

- Stock price (S) = f 100.--
- Exercise price (X) = f 125.--
- Risk-free interest rate (r_f) = 8% = 0.08
- Value per conversion right (W) = f 20.--

In the AM-approach the maturity (T) is estimated as the average maturity of the conversion rights, which is 3 years. Using the B/S-model and the earlier defined parameters, an ISD of 28.15% (0.2815) can be calculated.

In the PW-approach it is assumed that five warrants exist with maturities of 1, 2, 3, 4 and 5 years. Because the investor does not know which warrant he has got, he will assume a possibility of 20% for each warrant. This results in equation (25):

$$W = 0.20 \sum_{j=1}^5 W_j \quad (25)$$

where:

W = the value of a warrant (conversion right);

W_j = the value of part-warrant j.

The warrants have an average value of f 20.--. In the PW-approach we must find the ISD for which the value of the warrant-portfolio equals the value of the conversion right. In table 6 we have shown that if an ISD of 29.12% (0.2912) is assumed, five different warrant prices result with an average value of f 20.--.

Table 6: The PW-approach applied to the B/S-model.

	S	X	R_f	T	σ	W
warrant A	100	125	0.08	1	0.2912	6.26
warrant B	100	125	0.08	2	0.2912	13.82
warrant C	100	125	0.08	3	0.2912	20.64
warrant D	100	125	0.08	4	0.2912	26.83
warrant E	100	125	0.08	5	0.2912	<u>32.46</u>
						100.--

We notice that an important difference of both approaches is that the "term structure of volatility" is assumed to be flat, while in section 5.2.3 it is concluded that this is not the case in practice.

ad. 3) The possibility of conversion before the expiration date.

In case conversion is only possible at the expiration date of the bonds, the convertible bond holder has the right to choose between a repayment of the face value of the bonds or to exchange the bonds at their face value into shares of common stock at the conversion price. This situation is equal to an issue of bonds and European warrants, which have the same expiration date, the bonds are repaid at par and the warrants may be exercised at the payment of the exercise price.

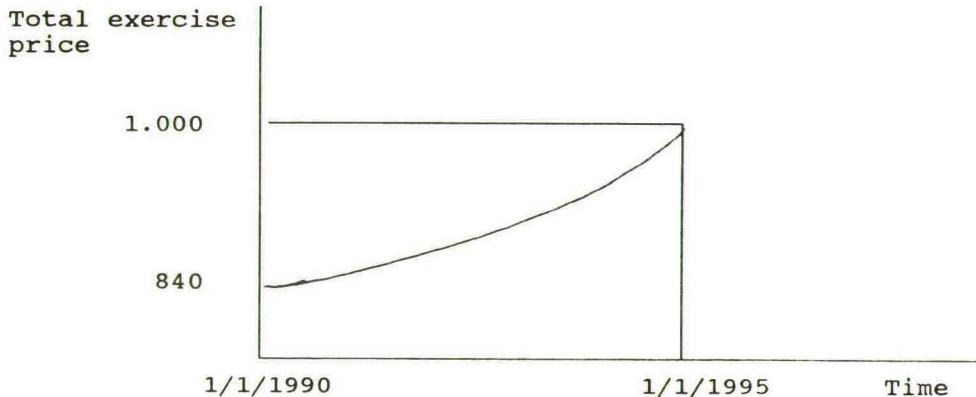
The situation is more complex in case conversion is possible before the expiration date. In order to clarify this we first define the 'formal' and the 'real' exercise price of the conversion right. The formal exercise price of a conversion right is the face value of the convertible bond. The real exercise price is the bond value of the convertible bond. The formal exercise price of the conversion right

remains constant during the maturity, this is not the case for the real exercise price. This is explained in example 5.

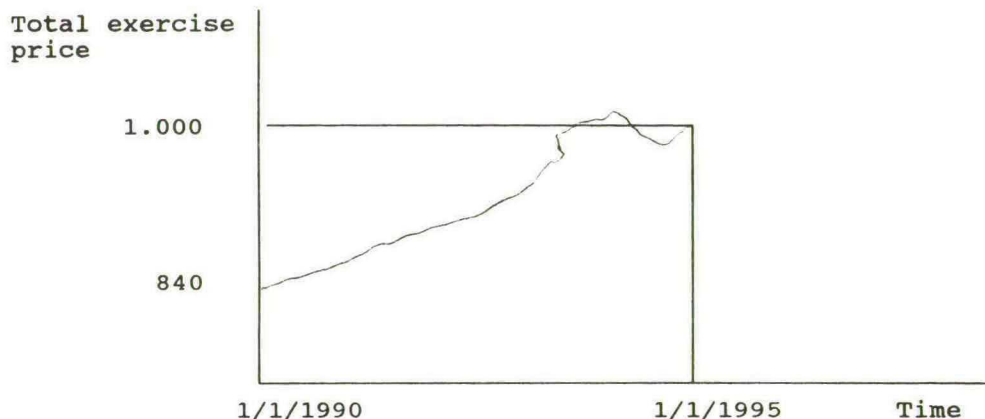
Example 5:

Consider the convertible bond of example 4. The convertible bond consisted of 8 warrants with a price of 20 guilders each, leading to a warrant-value of 160 guilders. Therefore the bond was (implicitly) issued at a price of 840 guilders. If the holder of the convertible bond wishes to convert immediately at January 1, 1990 he delivers his bond as a payment for the exercise price. In fact he pays 840 guilders as the exercise price for the conversion rights. If the holder of the convertible bond converts at the expiration date, he delivers a bond with the same face value, but with a different bond value, the latter is 1000 guilders instead of 840 guilders. Therefore the exercise price of the conversion right would increase in time from 840 guilders to 1000 guilders. This is presented in graph 1. If the interest rate changes, the exercise price still starts at 840 guilders and still ends at 1000 guilders, but the path of the exercise price changes over time. A possible path of the exercise price under a fluctuating interest rate is presented in graph 2.

Graph 1: the total exercise price of the conversion rights of examples 4 and 5 over time if the interest rate is assumed to be constant.



Graph 2: A possible path for the total exercise price of the conversion rights of examples 4 and 5 if the interest rate changes over time.



ad. 4) A fixed exchange rate in case of conversion.

The fourth problem that may occur is that a convertible bond, denominated in another currency than the underlying shares of common stock, is only convertible at a fixed exchange rate. As an example we mention the convertible bonds issued by the Dutch company Akzo in 1969. The bonds have a par value of \$ 1000. Holders of these convertible bonds could convert these bonds into shares of common stock at a fixed rate of 1 \$ = f 3.60. Notice that the real exercise price of the conversion right is now also determined by the exchange rate of the dollar. If e.g. at the expiration date the exchange rate of the dollar is below f 3.60 it may be profitable to convert the bond into shares even if the formal exercise price of the conversion rights remains below the stock price.

ad. 5) The inclusion of the possibility to call the convertible bonds.

The last problem that may occur is that the convertible bond is callable, which makes a comparison of ISDs derived from conversion rights with ISDs derived from warrants not very useful. Even if both the warrant and the conversion right are

callable, a simple comparison is not possible. In case of the warrant, only the maturity of the warrants is terminated, while in case of the convertible also the bond is redeemed. Generally convertible bonds are issued at par. In examples 4 and 5 we have already seen that in case part of the issue-price is accounted for by the conversion right, the bond-part of the convertible bond is issued at a discount. The general case is that the bond will continue to be traded at a discount during the maturity of the convertible bond. This means that at the 'call-date', the holder of the convertible bond would not only have a disadvantage due to the maturity reduction of his warrant, but also an advantage because of the bond redemption.

In exceptional cases the market rate of interest drops below the coupon interest rate of the convertible bond. In that case the bond will be traded above par (see also the last part of graph 2). This means that at the 'call-date', the holder of the convertible bond has as additional disadvantage, besides the maturity reduction of his warrant, in the sense that his bond, which trades above par, is redeemed at its face value.

The problems described above, make the calculation of ISDs from conversion rights a very complicated issue, that involves some problems for which we have not yet found a solution. This is especially true for the problems specified under points 3, 4 and 5. Even if a solution is found for all separate problems, the calculation of ISDs from the "implicit warrants" would probably be biased due to the great number of corrections to be made. This would make a comparison with ISDs from warrants and long term call-options not very useful.

10. Summary and conclusions.

In this paper the pricing of warrants is studied. In principle warrants can be valued using option pricing models.

Two classes of option pricing models can be identified: ad hoc models, which generally have the form of "multiple regression warrant valuation models", and models based on the Black/Scholes (1973) option pricing model (the B/S-model).

If the B/S-model is used for the valuation of warrants problems may exist because: (1) some warrants have special exercise provisions, e.g. they may be callable and (2) the maturity of warrants is generally much longer than the maturity of call-options. We notice that on the European Options Exchange in Amsterdam also call-options with initial maturities of five years are traded. Of course, the problems that exist for the valuation of warrants due to the longer maturities of warrants are also relevant for the long term call-options. Based on the B/S-model a number of option pricing models have been developed in which one or more of the assumptions underlying the B/S-model were relaxed. In this paper models are discussed in which:

- 1) dividend payments are included; these models are the Merton (1973) model, the binomial model and the American constant variance model;
- 2) the possibility of early exercise due to discrete dividend payments can be included, i.e. the binomial model and the American constant variance model;
- 3) the assumption of a constant volatility is replaced by the assumption of a volatility that decreases as the stock price increases, i.e. the constant elasticity of variance model; this model has as a special case the 'square root model', which assumes that the volatility is inversely related to the square root of the stock value;
- 4) some of the special exercise provisions can be included; Longstaff (1990) presents both a model that can deal with the companies right to extend the warrant's maturity and a model that can deal with the inclusion of a 'step-up-exercise price'.

Important parameter of all these models is the standard

deviation of the return on the underlying stock (from now on the standard deviation). This variable is not directly observable. In this paper two estimation procedures have been presented: the standard deviation based on historical stock price returns (historical standard deviation) and the implied standard deviation (ISD), which is the standard deviation that results if the market price of the option is equated to its model price. Empirical tests have shown that the implied standard deviation is a better predictor of the future standard deviation than the historical standard deviation is.

Important difference between a warrant and a call-option is, that when a call-option is exercised, only an exchange of existing shares from one market participant to another takes place. If warrants are exercised, new shares are created. This leads to an additional problem for the valuation of warrants, known as the 'dilution problem'.

The first solution presented for the dilution problem was to multiply the 'dilution factor' by the value of a call-option written on the stock of a firm without warrants. The dilution factor can be represented as $(1/(1+q))$ in which q represents the quotient of the number of new shares to be issued upon warrant exercise and the number of existing shares. This approach can only be used in case a firm issues warrants and uses the proceeds as a dividend payment to existing shareholders.

However, this approach can not be used for outstanding warrants. Due to the fact that the stock price already reflects the potential dilution of warrant exercise, the value of a call-option written on a firm without warrants can not be calculated. Therefore Schulz and Trautmann (1989) suggest to calculate the value of outstanding warrants by assuming that warrants are options on the firm's equity, instead of options on the firm's common stock. Schulz and Trautmann (1989) also derive a valuation formula based on this approach. From a comparison of the valuation of warrants by simply using the B/S-model, without a dilution-correction,

and the correct valuation formula discussed above, they conclude that the bias of simply using the B/S-model is very small. Therefore they argue that a dilution-correction per se is not necessary.

A problem with both dilution-corrections discussed above, is that they assume that warrants are exercised in a large block. In finance literature it has been shown that in case warrants are held by a monopolist he may benefit from exercising his warrants sequentially instead of in a block. However, it has also been shown that the existing shareholders have several possibilities to neutralize these advantages. It is also shown that competitive warrant holders may be worse off under the possibility of sequential exercise. We notice however, that this only occurs if extreme assumptions are made, e.g. with regard to the dividend policy of the firm.

In this paper also a number of empirical tests of option pricing models for the valuation of warrants have been discussed. The most important tests will be summarized below. Noreen and Wolfson (1981) test the Merton (1973) model and the square root model for a sample of 52 US warrant, which are all at-the-money. Using a historical standard deviation they come to the conclusion that both models work equally well and that they (slightly) tend to overvalue warrants.

Stucki and Wasserfallen (1989) test a number of option pricing models for a sample of Swiss warrants. They also use a historical standard deviation. Their conclusion is that even the B/S-model tends to undervalue warrants. This conclusion is remarkable because the B/S-model does not include dividend payments and is therefore to be expected to overvalue warrants rather than to undervalue them.

Schulz and Trautmann (1989) test the B/S-model corrected for dividend payments, the American constant variance model and the constant elasticity of variance model for a sample of German warrants. Also using historical standard deviations they conclude that the dividend corrected version of the B/S-

model is outperformed by both the American constant variance model and the constant elasticity of variance model. Both the American constant variance model and the constant elasticity of variance model neither overestimate nor underestimate warrant prices. With regard to the constant elasticity of variance model, Schulz and Trautmann (1989) conclude that the elasticity factor is highly unpredictable, therefore they doubt the superiority of this model compared to the American constant variance model.

Lauterbach and Schultz (1990) test a dividend corrected version of the B/S-model against a version of the square root model, corrected for dividends in the same way as the B/S-model. They use implied standard deviations calculated over quarter_t as an estimate of the standard deviation in quarter_{t-1}. Their conclusion is that the square root model outperforms the B/S-model.

We conclude this paper by presenting some interesting topics for the valuation of Dutch warrants:

- 1) the comparison of ISDs derived from warrant-, covered warrant- and call-option prices; on the basis of the research presented in this paper, a difference between ISDs should be related to:

- special warrant conditions;
- model misspecifications;

we notice that the calculation of ISDs from conversion rights may include a great number of additional problems; for some of the problems a solution has been presented, for other problems a solution is not yet available; notice however, that even if a solution is found for all individual problems, due to the great number of corrections that has to be made, the calculation of ISDs from conversion rights is likely to be biased;

- 2) the calculation of warrant prices using historical standard deviations; after the calculation of the relative differences between model- and market prices, a comparison with the valuation of US-, Swiss- and German warrants, as

presented by respectively Noreen and Wolfson (1981), Stucki and Wasserfallen (1989) and Schulz and Trautmann (1989) is possible;

3) interviews with investors in warrants;

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Appendix A: transaction costs for warrants and call-options.

In this appendix the transaction costs per share to be purchased by a private person are calculated for (1) buying the common stock directly or (2) buying and exercising call-options or warrants, and later selling the excess amount of shares to be purchased if the call-options or warrants are exercised. The calculations in this appendix are based on a brochure concerning transaction costs for securities and option transactions by private persons, issued by the Rabobank (one of the biggest three Dutch banks) in January 1989. The exact title of the brochure is: "Tarieven effecten- en optietransacties voor particulieren". A precise calculation is (on request) available from the author of this paper.

Example A1:

- stock price at the expiration date is f 50;
- exercise price of the call-option and the warrant is f 40;
- contract size of the warrant is 10 shares;
- contract size of the call-option is 100 shares (standard contract size at the EOE);
- purchase price per right to buy one share of common stock is f 7.50 for both warrant and call-option.

Table A1: Transaction costs per share of common stock purchased by the use of shares of common stock, call-options and warrants.

Number of shares to be purchased	Transaction costs shares	Transaction costs call-options	Transaction costs warrants
1	f 38.08	f 236.18	f 123.48
10	f 4.31	f 23.11	f 8.04
50	f 1.31	f 4.17	f 2.03
100	f 0.94	f 1.43	f 1.28
500	f 0.64	f 0.76	f 0.68

From table A1 we conclude that in case of call-options and warrants the transaction costs are extremely high in case only one share of common stock is to be purchased. This is due to the earlier mentioned fact that also transaction costs have to be made to sell the extra shares. In case of the call-option 99 shares have to be sold (100 shares are purchased if the contract is exercised and only one share is needed), in case of the warrant only 9 shares have to be sold. These transaction costs also occur for the call-option in case 10 or 50 shares have to be obtained. The difference

in transaction costs declines if a large number of shares is to be purchased (i.e. ≥ 500).

Appendix B: the Cox and Rubinstein model.

In addition to the verbal description of the Cox and Rubinstein model (see example 2 in sub-section 7.4) we present the equations of this model.

In table B1 the notation for the present value of each warrant conditional on the policy followed by both warrants is presented:

Table B1: Notation for the value of warrants.

	Both warrants are exercised at t=0	Only one war- rant is exer- cised at t=0	None of the warrants is exercised at t=0
Value of exercised warrant	A	B	not possible
Value of un- exercised warrant	not possible	C	D

The binomial valuation procedure is applied to find the values of warrants A, B, C and D:

$$A = \left[\frac{1}{N+2} \right] V - \left[\frac{N}{N+2} \right] X;$$

$$B = \left[\frac{1}{N+1} \right] V - \left[\frac{N}{N+1} \right] X - \left[\frac{1}{N+1} \right] C;$$

$$C = \left[\frac{p}{1+r_f} \right] \max \left[u \left[\frac{1}{N+2} \right] [V+X] - \left[\frac{N+1}{N+2} \right] X, 0 \right] + \\ \left[\frac{1-p}{1+r_f} \right] \max \left[d \left[\frac{1}{N+2} \right] [V+X] - \left[\frac{N+1}{N+2} \right] X, 0 \right]$$

$$D = \left[\frac{p}{1+r_f} \right] \max \left[u \left[\frac{1}{N+2} \right] V - \left[\frac{N}{N+2} \right] X, 0 \right] + \\ \left[\frac{1-p}{1+r_f} \right] \max \left[d \left[\frac{1}{N+2} \right] V - \left[\frac{N}{N+2} \right] X, 0 \right]$$

where:

V = the value of the firm's assets before warrants are exercised;

$p = (1+r_f-d)/(u-d)$.

The parameter values used in examples 2 and 3 lead to the following values for A, B, C and D.

	<u>Example 2</u>	<u>Example 3</u>
A =	f 149.33	f 149.33
B =	f 148.74	f 179.20
C =	f 152.28	f 0
D =	f 150.44	f 150.44

These variables are presented in an alternative way in tables 3 and 4.

Appendix C: the use of anti-dilution clauses for warrants and call-options.

In section 4.2 it has already been mentioned that warrants and call-options are both protected against the payment of large stock dividends (and/or the granting of preemptive rights and/or the distribution of bonus shares to existing shareholders). This protection is settled in an anti-dilution clause. Veld (1989a) argues that full protection against a decline in the 'theoretical bottom-value'⁴⁷ of a warrant requires that both the exercise price is lowered and the warrant-ratio is raised according to a specific formula. In a research after the use of anti-dilution clauses in Dutch warrant-agreements he concludes that the holders of these warrants are generally insufficiently protected against the payments of large stock-dividends, because the exercise price is lowered while the warrant-ratio remains the same⁴⁸.

The protection of call-options traded on the European Options Exchange (EOE) is only vaguely addressed to in the "Rules and Regulations" of the EOE. Therefore we have investigated the correction applied by the EOE for two large issues of bonus-shares: the 10%-issues of bonus-shares by the Dutch companies Ahold (in 1987) and KNP (in 1988). In both cases the exercise price was lowered and the option ratio was increased in such a way that the 'theoretical bottom value' remained constant. We notice that on the other hand, from casual empiricism it can be concluded that in case of small stock-dividends, anti-dilution clauses are applied for warrants, while they are not applied for EOE call-options.

Therefore our opinion is that the application of anti-dilution clauses will probably not be responsible for (large) differences between the prices of call-options and warrants.

Notes:

1. The author wishes to acknowledge the helpful comments and suggestions of drs. P.J.W. Duffhues, prof. dr. P.W. Moerland and drs. A.H.F. Verboven. Of course, all remaining errors are his own responsibility.

2. Part of this paper is derived from material earlier published. This is the case for parts of:

- section 5.2, see Veld (1989c) and Veld (1990);
- section 9.1, see Veld (1989b) and Veld and Verboven (1990a);
- section 9.2, see Veld and Verboven (1990b and 1991a).

3. In this paper a warrant is defined as: "A right issued by a company to buy a certain number of new shares in this company during a specific period (the exercise period) at a specific price (the exercise price)". This is in fact the definition of an equity call-warrant. We notice that also warrants are outstanding that give the right to sell the underlying value, so-called put-warrants, and warrants that give the right to buy or sell debt-securities, respectively called debt call-warrants and debt put-warrants. See Duffhues (1990) for a discussion of alternative warrant-types.

4. These 46 warrants include only the warrants issued by Dutch companies (including companies settled on the Netherlands Antilles) that have been listed on the "Officiële Markt" or the "Parallelmarkt" of the ASE. Besides these 46 warrants, also warrants may have been issued "Over-The-Counter".

5. In this paper the terms "pricing" and "valuation", as well as "price" and "value" are used as synonyms.

6. See Bick (1987) for a discussion on this matter.

7. See Black and Scholes (1973, page 640).

8. Formally we notice that the riskless interest rate is not directly observable. However, it can easily (and accurately) be estimated as the return on a government bond with the same maturity as the call-option. Besides that, the Black/Scholes (1973) model is not very sensitive to a change in the riskless interest rate, see e.g. Jarrow and Rudd (1983, pages 117-121).

9. In any case (equity-)call-options traded on the EOE are of the American type.

10. These warrants are generally referred to as CD-warrants (see e.g. Cremers (1979, page 86).

11. Longstaff (1990) mentions the following reasons for an extension of the warrant's maturity:

- US tax law includes a provision that in case a warrant expires unexercised, the price received upon issue is

taxable as income; if warrants are exercised this is not the case;

- if warrants expire unexercised, the firm faces substantial transaction costs associated with an equity-offering; however, if the warrants are extended and subsequently expire in-the-money, new shares can be issued at a price (probably) close to the market price with little or no marginal cost.

Notice that these reasons may also lead a company to include the right to reduce the exercise price on a temporary or definitive basis.

12. The return in the other time intervals is still $u_i = \ln(S_i/S_{i-1})$.

13. Calculated on the basis of the most recent 180 calendar days. This is the period from July 24, 1989 to January 19, 1990, which includes 125 observations ($n = 125$). r is calculated using trading days (one year is 254 trading days).

14. In-the-money call-options are call-options with an exercise price lower than the currently prevailing market price of the stock. At-the-money call-options have an exercise price which is equal to the stock price. Out-of-the-money call-options have an exercise price that is higher than the stock price.

15. See e.g. Latané and Rendleman (1976), Schmalensee and Trippi (1978), Chiras and Manaster (1978), Beckers (1981), Van der Hilst (1982 and 1989, pages 108-114), Brenner and Galai (1984) and Gemmill (1986).

16. This scheme is (partly) adopted from Gemmill (1986).

17. The partial derivative from the option price to the standard deviation for the B/S-model can be represented as:

$$\delta C / \delta \sigma = SN'(d_1) \sqrt{(T-t)}$$

where:

$$N'(d_1) = \frac{1}{\sqrt{(2\pi)}} e^{-d_1^2/2}$$

18. The weighing scheme used by Latané and Rendleman (1976) is reported in error in their original paper. The correct version of this weighing scheme can be found in: The Journal of Finance, 1979, page 1083.

19. The weighing scheme used by Beckers (1981, page 369) to compute the BISD comes down to minimizing the weighted sum of the squared deviations between the market value and the corresponding B/S-price.

20. Beckers (1981) uses the following rules in order to select a specific transaction price:

- the transaction price had to occur at least one hour before the market closed;
- the option price chosen was either the first trading price or the average of the first bid-ask quote after the stock price had changed.

21. The constant elasticity of variance model will be discussed in section 6.5.

22. For a more elaborate discussion of this model we refer to e.g. Cox, Ross and Rubinstein (1979), Jarrow and Rudd (1983, pages 175-197), Cox and Rubinstein (1985), Van der Hilst (1986 and 1989, pages 92-104), Kemna (1988, pages 11-14 and 77-80) and Verboven (1989).

23. The stock price at the end of the second period will either be u_uS , u_dS or ddS .

24. See e.g. Verboven (1989).

25. This approach is much like the Roll-Geske-Whaley model, see Roll (1977), Geske (1979) and Whaley (1981).

26. For a more elaborate discussion of this model we refer to e.g. Cox and Ross (1976), Beckers (1980) and Jarrow and Rudd (1983, pages 153-163).

27. According to Schulz and Trautmann (1989) the standard deviation (σ) may e.g. be estimated as the standard deviation implied in market prices for call-options.

28. For the intuition behind this we refer to e.g. Cox, Ross and Rubinstein (1979). They demonstrate the equality between (1) a portfolio containing a specific amount of call-options short and (2) a portfolio containing a specific amount of shares of common stock short and bonds long.

29. See e.g. Emanuel (1983), Constantinides and Rosenthal (1984), Constantinides (1984), Cox and Rubinstein (1985, pages 392-399), Ingersoll (1987, pages 435-445) and Spatt and Sterbenz (1988).

30. According to Emanuel (1983) the situation of a monopolist can arise in a number of ways, of which he mentions:

- 1) an issue of warrants by the senior management to themselves in order to avoid an unwelcome take-over;
- 2) an issue of warrants to a venture capitalist who has provided substantial financing;
- 3) an issue of warrants to an underwriter as an additional form of compensation.

31. This example is based on a model developed by Cox and Rubinstein (1985, pages 396-399). Although the example presented by Cox and Rubinstein (1985) illustrates the second possibility mentioned by Spatt and Sterbenz (1988), we argue that a large number of simulations with this model led us to the conclusion that the increase in value of the unexercised warrants generally does not exceed the forfeiture on the premium above parity for the exercised warrants. Therefore we emphasize that use of the funds to expand the firm's investment policy may lead to sequential exercise instead of block exercise, however this will certainly not always be the case.

32. For an explicit proof of these possibilities, see Spatt and Sterbenz (1988).

33. Any excess funds are to be paid out to shareholders as an extra dividend.

34. Competitive warrant holders under the possibility of sequential exercise are only worse off if $B > C$ and at the same time $A > D$. Furthermore we remark that Cox, Ross and Rubinstein (1979, page 232) prove that, in order to avoid riskless arbitrage possibilities, the binomial model requires that $u > 1 + r_f > d$. This means that a situation where $B > C$ and $A > D$ only occurs if $r_f < 0$. This is not possible because r_f represents the nominal rate of interest, not the real rate of interest. (See appendix B for a definition of the symbols used in this note).

35. For a discussion of the cases between a monopolist and competitive warrant holders, such as the case for oligopoly warrant holders we refer to Constantinides and Rosenthal (1984), Constantinides (1984) and especially to Spatt and Sterbenz (1988).

36. The equation for the Shelton (1967b) model is presented by Shelton (1967b) and Noreen (1982).

37. Specifically the absolute value of the ratio $R (= (S-X)/X)$ is not larger than 20%. From the 52 warrants, only 5 had a ratio R between 10 and 20 percent. The other 47 warrants had a ratio R smaller than 10%.

38. The exact equation for the square root model including a continuous dividend payment can be found in Cox and Ross (1976, page 161) and Noreen and Wolfson (1981, page 388).

39. As other possible factors responsible for the lower prediction error we mention:

- the estimation of the riskless interest rate is slightly different, however, as we have mentioned before the Merton model is not very sensitive to a change in the riskless interest rate;
- the dilution factors, for which erroneously is corrected,

may have been larger in the Noreen (1982) study than in the Noreen and Wolfson (1981) study; we notice that this explanation is purely speculative because neither study provides information about the level of the dilution factors.

40. We clarify the nature of such a provision by an example: in case a warrant is exercised in 1987, it is first entitled to the dividend over 1987, which will be paid in 1988 (assuming that there are no interim-dividends). The dividend paid in 1987 is the dividend over 1986, to which the former warrant holders are not entitled. See Schulz and Trautmann (1989, pages 18-19) for the exact correction-formula.

41. See Christie (1982) for an exact description of the maximum likelihood procedure.

42. Also a number of other factors which could a priori be expected to be responsible for this inverse relationship were tested. These factors were:

- 1) equity volatility is stochastic but uncorrelated with equity values;
- 2) improper dividend adjustments;
- 3) ignoring the possibility of extension;
- 4) non-synchronous trading.

However, Lauterbach and Schultz (1990) prove that these factors are not responsible for the inverse relationship.

43. 'Covered warrants', also known as 'falcons' may also be included in the analysis. As an example we mention the falcons issued by Robeco, in corporation with Arab Banking Corporation, to buy existing shares of 'Koninklijke Olie'. These covered warrants are in fact call-options, which are not traded on an official options exchange, but on the stock exchange.

44. See e.g. Weston and Copeland (1986, page 850) and Hull (1989, pages 21 and 253).

45. As mentioned in note 2, this analysis is largely based on Veld and Verboven (1990b and 1991a). We also want to acknowledge the helpful comments of drs. P.N. Wijn (Robeco) on this subject.

46. A possible method is suggested by Veld and Verboven (1990b).

47. The theoretical bottom value, also known as the intrinsic value, is the difference between the currently prevailing price of the underlying common stock and the exercise price ($S-X$), multiplied by the warrant-ratio.

48. In this context it is interesting to remark that Stucki and Wasserfallen (1989) argue that this is also the case for Swiss warrants. Notice also the quotation of Spatt and

Sterbenz (1988) in section 7.4 regarding the latitude the firm enjoys regarding anti-dilution protection.

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